A Multivariate Student-t Process Model for Dependent Tail-weighted Degradation Data

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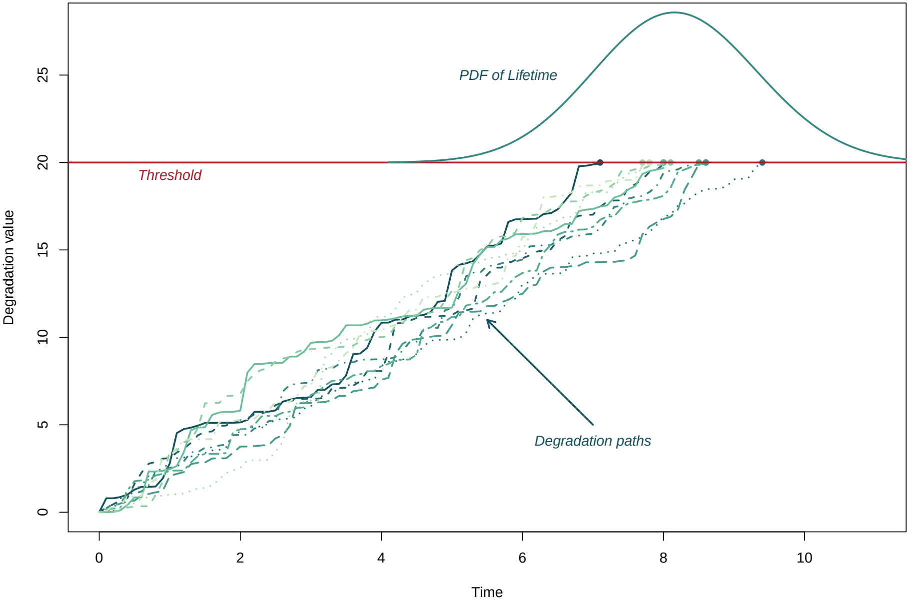
Outline


PDF of Lifetime
Threshold
Degradation paths
Degradation value
Time
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5
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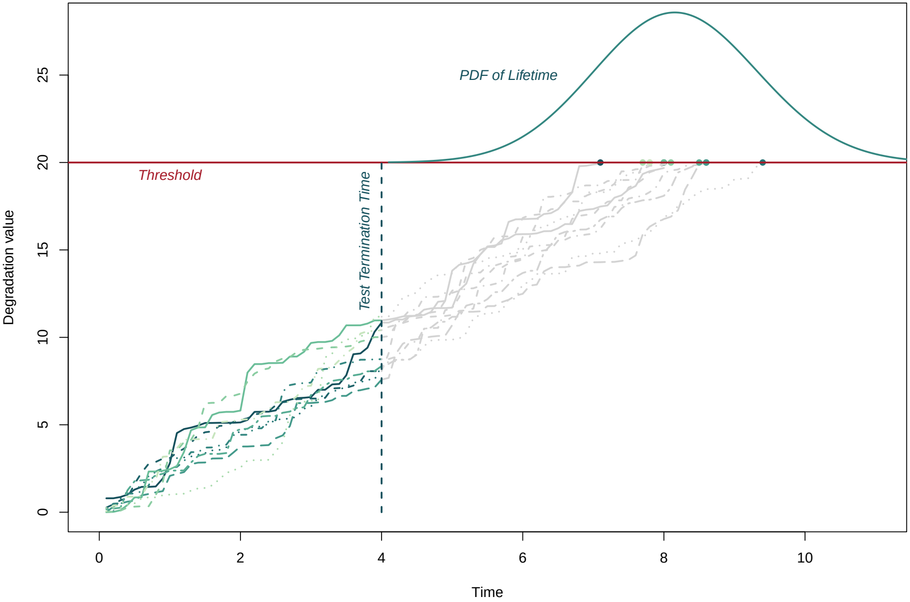
PDF of Lifetime
Threshold
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Degradation value
Time
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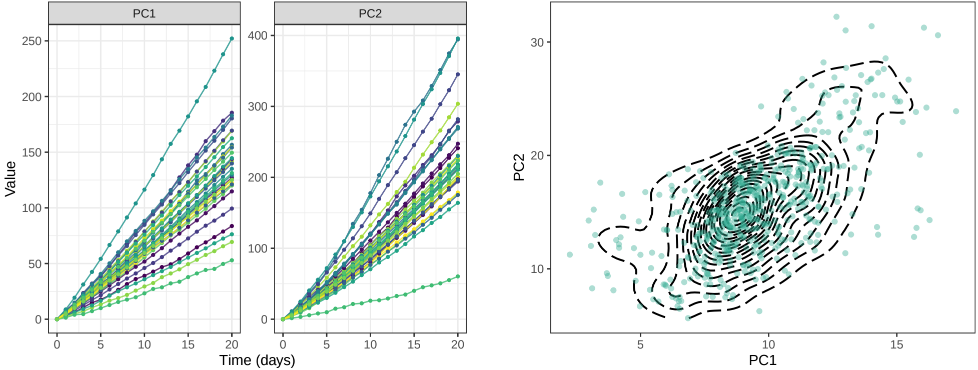
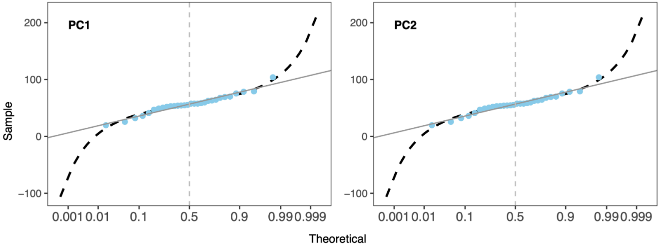
Motivation



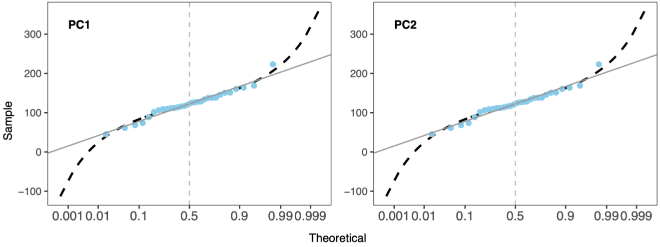
(a) Degradation paths

(b) Contour plots

Figure 1: Permanent magnet brake (PMB) dagradation data.



(a) $t = 9$



(b) $t = 18$

Figure 2: Normal Q-Q plot of PMB data, where the black dotted line is the Student's t distribution, and the grey solid line is the normal distribution.

Related Literature

Degradation performance characteristic (PC) modeling

1. Single PC: **general path models** and **stochastic process models** (Wiener process, Gamma process, Inverse Gaussian process)
2. Two or more PCs:
   (i) Copula-based method: (Fang et al., 2020; Sun et al., 2021; Zhuang et al., 2021).
   (ii) Multivariate distribution-based method: (Fang and Pan, 2023; Pan and Balakrishnan, 2011).
   (iii) Common-effect-based method: (Liu et al. 2021; Zhai and Ye 2023).
3. Limitations: Accommodate the heavy-tailed characteristics.

Contributions

(i) A tail-weighted multivariate process to characterize the degradation processes of multiple dependent PCs.

(ii) Derive the lifetime distribution and propose a Monte Calor method to estimate the reliability function.

(iii) Present a two-stage method involving nonlinear least squares (NLS) estimation and an expectation maximization (EM) algorithm, followed by the utilization of a bootstrap approach for interval estimation.

Outline

A new multivariate degradation model

A system with p PCs, $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_p(t))'$ is degradation value.

Model

$$\begin{cases} Y_j(t) = \theta_j \Lambda_j(t) + \dfrac{\delta_j}{\sqrt{\tau}} W_j(\Lambda_j(t)), j = 1, \ldots, p, \\ \boldsymbol{\Theta} = (\theta_1, \theta_2, \ldots, \theta_p)' \sim \mathcal{N}_p(\boldsymbol{\eta}, \boldsymbol{\Sigma}_0/\tau), \\ \tau \sim \mathcal{G}(\nu/2, 2/\nu), \end{cases} \tag{1}$$

1. θ_j is drift parameter, δ_j is diffusion parameter;
2. $\Lambda_j(t)$ is time scale transformation function (non-negative, monotonically increasing).
3. $W_i(\cdot)$s are independent standard Brownian motions.
4. $\eta_j > 0$, and $\boldsymbol{\Sigma}_0 = (\sigma_{ij})_{p \times p}$ is a positive definite matrix.
5. τ follows a gamma distribution.

A new multivariate degradation model

Let $\boldsymbol{\Sigma}(t) = \text{diag}\{\Lambda_1(t), \dots, \Lambda_p(t)\}$ and $\boldsymbol{\Omega}_\delta = \text{diag}\{\delta_1^2, \dots, \delta_p^2\}$.

Joint distribution of $\boldsymbol{Y}(t)$

$$\boldsymbol{Y}(t) \sim \mathcal{T}_p\left(\boldsymbol{\Lambda}_\eta(t), \boldsymbol{U}(t), \nu\right), \tag{2}$$

where $\mathcal{T}_p(\cdot, \cdot, \nu)$ denotes the p-dimensional Student's t distribution with ν degrees of freedom, $\boldsymbol{\Lambda}_\eta(t) = (\eta_1\Lambda_1(t), \dots, \eta_p\Lambda_p(t))'$ and $\boldsymbol{U}(t) = \boldsymbol{\Sigma}(t)\boldsymbol{\Sigma}_0\boldsymbol{\Sigma}(t) + \boldsymbol{\Sigma}(t)\boldsymbol{\Omega}_\delta$.

Remark: when $\nu = 1$, it becomes a multivariate Cauchy distribution; when $\nu \to \infty$, it reduces to a multivariate normal distribution.

A new multivariate degradation model

- Assume m measurements for the j-th PC with $\boldsymbol{t}(m) = (t_1, \ldots, t_m)'$.
- $Y_j(\boldsymbol{t}_{(m)}) = (Y_j(t_1), \ldots, Y_j(t_m))'$ are degradation values of the j-th PC.

Joint distribution of $Y_j(\boldsymbol{t}_{(m)})$

$$Y_j(\boldsymbol{t}_{(m)}) \sim \mathcal{T}_m\left(\eta_j \Lambda_j(\boldsymbol{t}_{(m)}), V_j(\boldsymbol{t}_{(m)}), \nu\right), \tag{3}$$

- $\Lambda_j(\boldsymbol{t}_{(m)}) = (\Lambda_j(t_1), \ldots, \Lambda_j(t_m))'$,
- $Q(\boldsymbol{t}_{(m)}) = [\min\{\Lambda_j\left(t_{s_1}\right), \Lambda_j\left(t_{s_2}\right)\}]_{1 \leq s_1, s_2 \leq m}$.
- $V_j(\boldsymbol{t}_{(m)}) = \sigma_j^2 \Lambda_j(\boldsymbol{t}_{(m)}) \Lambda_j(\boldsymbol{t}_{(m)})' + \delta_j^2 Q(\boldsymbol{t}_{(m)})$.
- σ_j^2 is the j-th element on the diagonal of the matrix $\boldsymbol{\Sigma}_0$.

Reliability analysis

- ω_j denote the failure threshold level for the j-th PC.
- Lifetime of the j-th PC is $T_j = \inf\{t : Y_j(t) \geq \omega_j$.
- With θ_j, τ, $Y_j(t)$ follows a Wiener process, yielding $\Lambda_j(T_j) \sim \mathcal{IG}(\omega_j/\theta_j, \omega_j^2\sqrt{\tau}/\delta_j)$.

Conditional pdf of T_j

$$f_j(t|\theta_j, \tau) = \frac{\omega_j}{\sqrt{2\pi\delta_j^2\Lambda_j^3(t)/\tau}} \exp\left\{-\frac{(\omega_j - \theta_j\Lambda_j(t))^2\tau}{2\delta_j^2\Lambda_j(t)}\right\} \frac{\mathrm{d}\Lambda_j(t)}{\mathrm{d}t}. \quad (4)$$

Joint pdf for $T_1, T_2, \ldots, T_p$

$$f(t_1, t_2, ..., t_p) = \int\int \prod_{j=1}^{p} f_j(t_j|\theta_j, \tau) f(\boldsymbol{\Theta}|\tau) f(\tau) \mathrm{d}\boldsymbol{\Theta}\mathrm{d}\tau. \quad (5)$$

Reliability analysis

- Assume system to have failed when any PC reaches the failure threshold level.

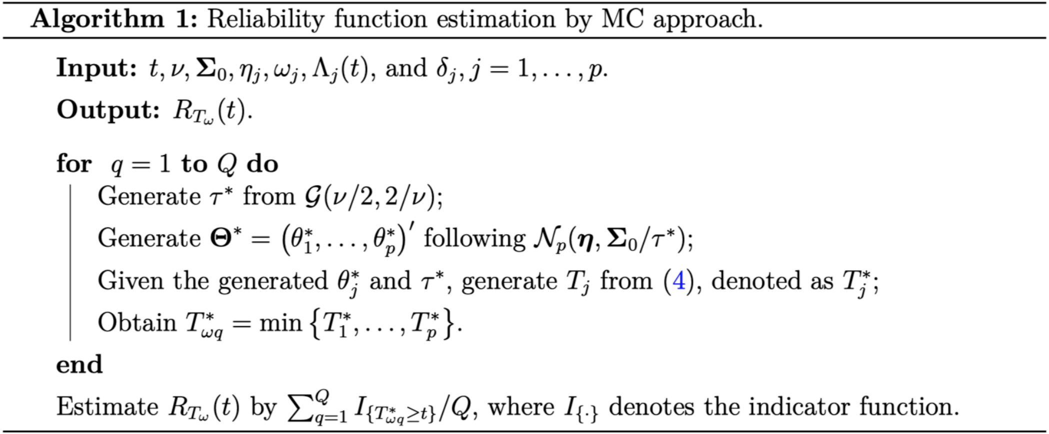
System lifetime

$$T_\omega = \inf\{t : Y_1(t) \geq \omega_1 \text{ or } \cdots \text{ or } Y_p(t) \geq \omega_p\} = \min\{T_1, \ldots, T_p\}. \tag{6}$$

System reliability

$$\begin{aligned} R_{T_\omega}(t) &= P\{T_\omega > t\} = P\{T_1 > t, \ldots, T_p > t\} \\ &= \int_t^{+\infty} \cdots \int_t^{+\infty} f(t_1, t_2, ..., t_p)\mathrm{d}t_1 \cdots \mathrm{d}t_p. \end{aligned} \tag{7}$$

Reliability analysis



Algorithm 1: Reliability function estimation by MC approach.

```
Input: $t, \nu, \boldsymbol{\Sigma}_0, \eta_j, \omega_j, \Lambda_j(t)$, and $\delta_j, j = 1, \dots, p$.
Output: $R_{T_\omega}(t)$.

for $q = 1$ to $Q$ do
    Generate $\tau^*$ from $\mathcal{G}(\nu/2, 2/\nu)$;
    Generate $\boldsymbol{\Theta}^* = \left(\theta_1^*, \dots, \theta_p^*\right)'$ following $\mathcal{N}_p(\boldsymbol{\eta}, \boldsymbol{\Sigma}_0/\tau^*)$;
    Given the generated $\theta_j^*$ and $\tau^*$, generate $T_j$ from (4), denoted as $T_j^*$;
    Obtain $T_{\omega q}^* = \min\left\{T_1^*, \dots, T_p^*\right\}$.
end
Estimate $R_{T_\omega}(t)$ by $\sum_{q=1}^{Q} I_{\{T_{\omega q}^* \geq t\}}/Q$, where $I_{\{\cdot\}}$ denotes the indicator function.
```

Outline

Statistical inference

- Assume n systems in an experiment, degradation is measured at $t_{i,1}, \ldots, t_{i,m_i}$,
- Degradation values at time $t_{i,k}$ are $\boldsymbol{Y}_{i,k} = (Y_{i,1,k}, \ldots, Y_{i,p,k})', i = 1, \ldots, n$ and $k = 1, \ldots, m_i$.
- Let $\Delta\boldsymbol{Y}_{i,k} = \boldsymbol{Y}_{i,k} - \boldsymbol{Y}_{i,k-1}$, where $t_{i,0} = 0$ and $\boldsymbol{Y}_{i,0} = \mathbf{0}$.

Model

$$\begin{cases} \Delta\boldsymbol{Y}_{i,k}|\boldsymbol{\Theta}_i, \tau_i \sim \mathcal{N}_p\left(\Delta\boldsymbol{\Sigma}(t_{i,k})\boldsymbol{\Theta}_i, \dfrac{\boldsymbol{\Omega}_\delta}{\tau_i}\Delta\boldsymbol{\Sigma}(t_{i,k})\right), \\ \boldsymbol{\Theta}_i \sim \mathcal{N}_p(\boldsymbol{\eta}, \boldsymbol{\Sigma}_0/\tau_i), \\ \tau_i \sim \mathcal{G}(\nu/2, 2/\nu), \end{cases} \tag{8}$$

where $\Delta\boldsymbol{\Sigma}(t_{i,k}) = \boldsymbol{\Sigma}(t_{i,k}) - \boldsymbol{\Sigma}(t_{i,k-1})$, $\boldsymbol{\Sigma}(t) = \text{diag}\{\Lambda_1(t), \ldots, \Lambda_p(t)\}$.

Statistical inference

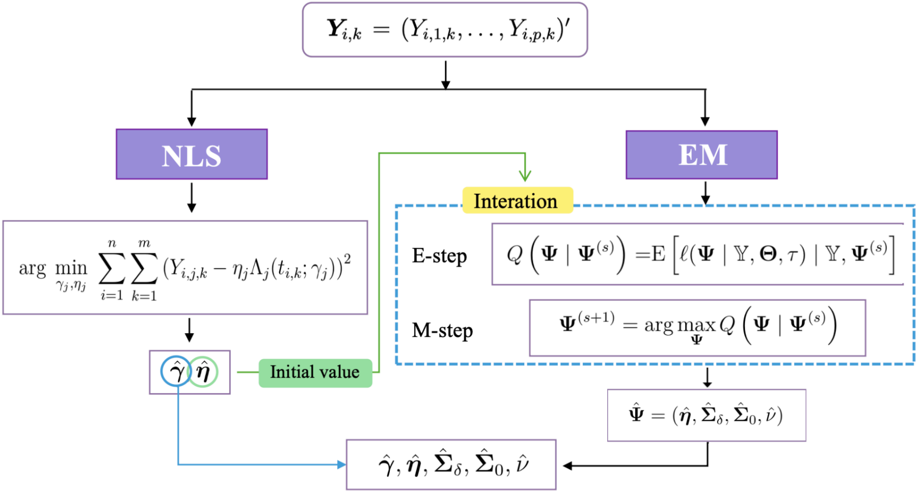
- For the j-th time scale transformation function $\Lambda_j(t)$, we assume a parametric form with an unknown parameter γ_j, represented as $\Lambda_j(t; \gamma_j)$.
- The choice of the specific form for $\Lambda_j(t; \gamma_j)$ can be determined based on engineering experience or empirical investigation.
- Commonly used forms include the power law function $\Lambda_j(t; \gamma_j) = t^{\gamma_j}$ and the exponential function $\Lambda_j(t; \gamma_j) = \exp(\gamma_j t) - 1$.
- Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_p)'$. Then the model parameters are $\boldsymbol{\Phi} = (\boldsymbol{\eta}, \boldsymbol{\Omega}_\delta, \boldsymbol{\Sigma}_0, \boldsymbol{\gamma}, \nu)$.

Statistical inference

Let $\Delta \boldsymbol{Y}_i = \{\Delta \boldsymbol{Y}_{i,k}, k = 1, \ldots, m_i\}$ and $\mathbb{Y} = \{\Delta \boldsymbol{Y}_i, i = 1, \ldots, n\}$.

Likelihood function of $\boldsymbol{\Phi}$

$$\begin{aligned} \ell(\mathbb{Y}|\boldsymbol{\Phi}) = & \prod_{i=1}^{n} \int\!\!\int \Big[\prod_{k=1}^{m_i} \frac{\tau_i^{p/2}}{(2\pi)^{p/2} \Big| \boldsymbol{\Omega}_\delta \Delta \boldsymbol{\Sigma}(t_{i,k}) \Big|^{1/2}} \\ & \times \exp\left\{ -\frac{\tau_i}{2} (\Delta \boldsymbol{Y}_{i,k} - \Delta \boldsymbol{\Sigma}(t_{i,k}) \boldsymbol{\Theta}_i)' \big(\boldsymbol{\Omega}_\delta \Delta \boldsymbol{\Sigma}(t_{i,k}) \big)^{-1} (\Delta \boldsymbol{Y}_{i,k} - \Delta \boldsymbol{\Sigma}(t_{i,k} \boldsymbol{\Theta}_i) \right\} \Big] \\ & \times \frac{\tau_i^{p/2}}{(2\pi)^{p/2} |\boldsymbol{\Sigma}_0|^{1/2}} \exp\left\{ -\frac{\tau_i}{2} (\boldsymbol{\Theta}_i - \boldsymbol{\eta})' \boldsymbol{\Sigma}_0^{-1} (\boldsymbol{\Theta}_i - \boldsymbol{\eta}) \right\} \\ & \times \frac{\tau_i^{\nu/2-1}}{\Gamma(\nu/2)(2/\nu)^{\nu/2}} \exp\left\{ -\frac{\nu}{2} \tau_i \right\} \mathrm{d}\boldsymbol{\Theta}_i \mathrm{d}\tau_i. \end{aligned} \tag{9}$$



$$\boldsymbol{Y}_{i,k} = (Y_{i,1,k}, \ldots, Y_{i,p,k})'$$

NLS

EM

$$\underset{\gamma_j,\eta_j}{\arg\min} \sum_{i=1}^{n}\sum_{k=1}^{m}\left(Y_{i,j,k} - \eta_j\Lambda_j(t_{i,k};\gamma_j)\right)^2$$

$\hat{\gamma}$ $\hat{\boldsymbol{\eta}}$

Initial value

Interation

E-step $Q\left(\boldsymbol{\Psi} \mid \boldsymbol{\Psi}^{(s)}\right) = \mathrm{E}\left[\ell(\boldsymbol{\Psi} \mid \mathbb{Y}, \boldsymbol{\Theta}, \tau) \mid \mathbb{Y}, \boldsymbol{\Psi}^{(s)}\right]$

M-step $\boldsymbol{\Psi}^{(s+1)} = \underset{\boldsymbol{\Psi}}{\arg\max}\, Q\left(\boldsymbol{\Psi} \mid \boldsymbol{\Psi}^{(s)}\right)$

$$\hat{\boldsymbol{\Psi}} = (\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\Sigma}}_\delta, \hat{\boldsymbol{\Sigma}}_0, \hat{\nu})$$

$$\hat{\gamma}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\Sigma}}_\delta, \hat{\boldsymbol{\Sigma}}_0, \hat{\nu}$$

Figure 3: Proposed two-stage algorithm for model parameter estimation.

Nonlinear least squares estimation

- NLS is a statistical method for parameter estimation in nonlinear regression models, achieved by **minimizing the sum of squared residuals** (SSR).
- $E[Y_j(t)] = \eta_j \Lambda_j(t;\gamma_j)$: Relationship between degradation values of j-th PC and t.
- Given $\{\mathbf{Y}_{i,k}, i = 1, \ldots, n, k = 1, \ldots, m_i\}$, SSR for the j-th PC is:

$$SSR_j = \sum_{i=1}^{n} \sum_{k=1}^{m} \left(Y_{i,j,k} - \eta_j \Lambda_j(t_{i,k};\gamma_j) \right)^2, j = 1, \ldots, p. \tag{10}$$

- The estimate $(\hat{\gamma}_j, \hat{\eta}_j)$ can be obtained by minimizing SSR_j, that is,

$$(\hat{\gamma}_j, \hat{\eta}_j) = \arg \min_{\gamma_j, \eta_j} SSR_j. \tag{11}$$

- Once $(\hat{\gamma}_j, \hat{\eta}_j), j = 1, \ldots, p$ are obtained, $\Lambda_j(t;\hat{\gamma}_j)$ is treated as a known function, and $\hat{\eta}_j$ is used as an initial value in the EM algorithm.

EM algorithm

- $(\boldsymbol{\Theta}, \boldsymbol{\tau}) = \{\boldsymbol{\Theta}_i, \tau_i, i = 1, \ldots, n\}$ as the missing data, $\Delta\Lambda_j(t_{i,k}) = \Lambda_j(t_{i,k}) - \Lambda_j(t_{i,k-1})$.

Log-likelihood function of $\boldsymbol{\Psi}$

$$\begin{aligned}\ell(\mathbb{Y}, \boldsymbol{\Theta}, \boldsymbol{\tau}|\boldsymbol{\Psi}) = &\sum_{i=1}^{n} \Bigg\{ \ell_c + \Big(\frac{(m_i+1)p+\nu}{2} - 1 \Big) \ln \tau_i - m_i \sum_{j=1}^{p} \ln \delta_j \\ &- \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{m_i} \ln \Delta\Lambda_j(t_{i,k}) \Bigg\} - \frac{1}{2} \sum_{i=1}^{n} \tau_i \left(\sum_{k=1}^{m_i} \ell_{i,k} + \ell_{i,0} + \nu \right),\end{aligned} \tag{12}$$

where

$$\begin{aligned}\ell_c &= -\frac{(m_i+1)p}{2} \ln 2\pi - \frac{1}{2} \ln |\boldsymbol{\Sigma}_0| - \ln \Gamma(\frac{\nu}{2}) + \frac{\nu}{2} \ln(\frac{\nu}{2}),\\ \ell_{i,0} &= (\boldsymbol{\Theta}_i - \boldsymbol{\eta})' \boldsymbol{\Sigma}_0^{-1} (\boldsymbol{\Theta}_i - \boldsymbol{\eta}),\\ \ell_{i,k} &= (\Delta \boldsymbol{Y}_{i,k} - \Delta\boldsymbol{\Sigma}(t_{i,k})\boldsymbol{\Theta}_i)' \big(\boldsymbol{\Omega}_\delta \Delta\boldsymbol{\Sigma}(t_{i,k})\big)^{-1} (\Delta \boldsymbol{Y}_{i,k} - \Delta\boldsymbol{\Sigma}(t_{i,k})\boldsymbol{\Theta}_i).\end{aligned}$$

Q-function

$$
\begin{aligned}
Q\left(\boldsymbol{\Psi} \mid \boldsymbol{\Psi}^{(s)}\right) =& \mathrm{E}\left[\ell(\boldsymbol{\Psi} \mid \mathbb{Y}, \boldsymbol{\Theta}, \tau) \mid \mathbb{Y}, \boldsymbol{\Psi}^{(s)}\right] \\
=& \sum_{i=1}^{n}\left\{\ell_c + \left(\frac{(m_i+1)p+v}{2} - 1\right) \mathrm{E}\left[\ln \tau_i | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right] - m_i \sum_{j=1}^{p} \ln \delta_j \right. \\
& \left. - \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{m_i} \ln \Delta\Lambda_j(t_{i,k}) \right\} - \frac{1}{2} \sum_{i=1}^{n} \left\{ \sum_{k=1}^{m_i} \mathrm{E}\left[\tau_i \ell_{i,k} | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right] \right. \\
& \left. + \mathrm{E}\left[\tau_i \ell_{i,0} | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right] + \nu \mathrm{E}[\tau_i | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}] \right\}.
\end{aligned}
\tag{13}
$$

Theorem 1

(a) $\boldsymbol{\Theta}_i|\Delta \boldsymbol{Y}_i, \tau_i \sim \mathcal{N}_p\left(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}/\tau_i\right)$, where $\boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i} = \left[\boldsymbol{\Sigma}_0^{-1} + \boldsymbol{\Omega}_\delta^{-1}\boldsymbol{\Sigma}(t_{im_i})\right]^{-1}$ and $\boldsymbol{\mu}_i = \boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\eta} + \boldsymbol{\Omega}_\delta^{-1}\boldsymbol{Y}_{i,m_i})$.

(b) $\tau_i|\Delta \boldsymbol{Y}_i \sim \mathcal{G}\left(\frac{m_i p+\nu}{2}, \frac{2}{K_{i,1}-K_{i,2}+\nu}\right)$, where $K_{i,1} = \sum_{k=1}^{m_i} \Delta \boldsymbol{Y}_{i,k}' \boldsymbol{\Omega}_\delta^{-1} \Delta\boldsymbol{\Sigma}^{-1}(t_{i,k})\Delta \boldsymbol{Y}_{i,k} + \boldsymbol{\eta}^\top \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\eta}$ and $K_{i,2} = \boldsymbol{\mu}_i' \boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}^{-1}\boldsymbol{\mu}_i$.

$\mathrm{E}(\tau_i|\Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)})$ and $\mathrm{E}(\ln \tau_i|\Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)})$

$$\begin{aligned} \mathrm{E}(\tau_i|\Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}) &= \frac{m_i p + \nu^{(s)}}{K_{i,1}^{(s)} - K_{i,2}^{(s)} + \nu^{(s)}}, \\ \mathrm{E}(\ln \tau_i|\Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}) &= \psi\left(\frac{m_i p + \nu^{(s)}}{2}\right) - \ln\left(\frac{K_{i,1}^{(s)} - K_{i,2}^{(s)} + \nu^{(s)}}{2}\right), \end{aligned} \tag{14}$$

where $\psi(x) = \mathrm{d}\ln\Gamma(x)/\mathrm{d}x$ is the digamma function, $K_{i,1}^{(s)}$ and $K_{i,2}^{(s)}$ are $K_{i,1}$ and $K_{i,2}$ with the parameters $\boldsymbol{\Psi}$ substituted by $\boldsymbol{\Psi}^{(s)}$.

Theorem 2

Given the joint distributions of $\boldsymbol{\Theta}_i$ and τ_i in Theorem 1, if the solution in the M-step at the s-th iteration is $\boldsymbol{\Psi}^{(s)}$, then

$$
\begin{aligned}
&\mathrm{E}\left(\tau_i \ell_{i,0} | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right) = \operatorname{tr}\left(\boldsymbol{\Sigma}_0^{-1} \boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}^{(s)}\right) + \mathrm{E}\left(\tau_i | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right) \left(\boldsymbol{\mu}_i^{(s)} - \boldsymbol{\eta}\right)' \boldsymbol{\Sigma}_0^{-1} \left(\boldsymbol{\mu}_i^{(s)} - \boldsymbol{\eta}\right), \\
&\mathrm{E}\left(\tau_i \ell_{i,k} | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right) = \operatorname{tr}\left(\Delta \boldsymbol{\Sigma}(t_{i,k}) \boldsymbol{\Omega}_\delta^{-1} \boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}^{(s)}\right) + \mathrm{E}\left(\tau_i | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right) \\
&\qquad \times \left(\boldsymbol{\mu}_i^{(s)} - \Delta \boldsymbol{\Sigma}^{-1}(t_{i,k}) \Delta \boldsymbol{Y}_{i,k}\right)' \left(\boldsymbol{\Omega}_\delta \Delta \boldsymbol{\Sigma}^{-1}(t_{i,k})\right)^{-1} \left(\boldsymbol{\mu}_i^{(s)} - \Delta \boldsymbol{\Sigma}^{-1}(t_{i,k}) \Delta \boldsymbol{Y}_{i,k}\right).
\end{aligned}
$$

- Given the results in theorems 1 and 2, the Q-function can be completely determined.
- Then we update the optimal solution in the M-step at the $(s+1)$-th iteration as

$$\boldsymbol{\Psi}^{(s+1)} = \arg\max_{\boldsymbol{\Psi}} Q\left(\boldsymbol{\Psi} \mid \boldsymbol{\Psi}^{(s)}\right). \tag{15}$$

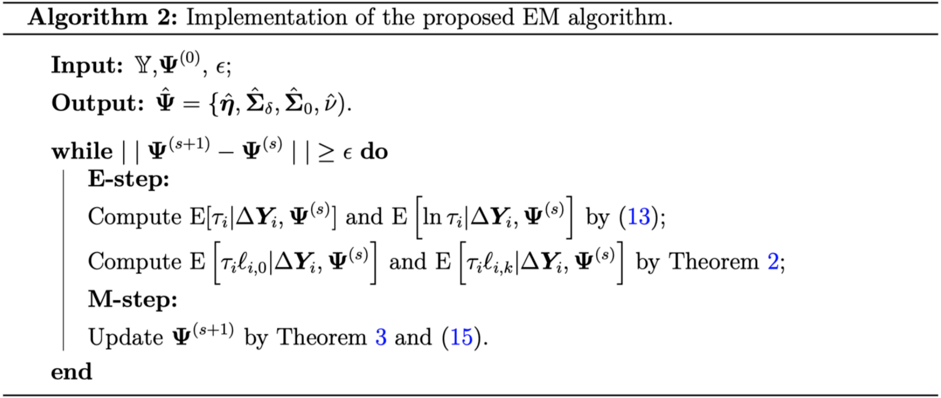
Theorem 3

Given the solution in the M-step at the s-th iteration is $\boldsymbol{\Psi}^{(s)}$, the solution of (15) can be updated as follows:

$$
\begin{aligned}
\boldsymbol{\eta}^{(s+1)} &= \frac{\sum_{i=1}^{n} \boldsymbol{\mu}_i^{(s)} \mathrm{E}\left(\tau_i | \boldsymbol{\Delta Y}_i, \boldsymbol{\Psi}^{(s)}\right)}{\sum_{i=1}^{n} \mathrm{E}\left(\tau_i | \boldsymbol{\Delta Y}_i, \boldsymbol{\Psi}^{(s)}\right)}, \\
\boldsymbol{\Sigma}_0^{(s+1)} &= \frac{\sum_{i=1}^{n} \left[\boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}^{(s)} + \mathrm{E}(\tau_i | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}) (\boldsymbol{\mu}_i^{(s)} - \boldsymbol{\eta}^{(s+1)}) (\boldsymbol{\mu}_i^{(s)} - \boldsymbol{\eta}^{(s+1)})' \right]}{n}, \\
\boldsymbol{\Omega}_\delta^{(s+1)} &= \frac{1}{\sum_{i=1}^{n} m_i} \sum_{i=1}^{n} \sum_{k=1}^{m_i} \Big[\Delta \boldsymbol{\Sigma}(t_{i,k}) \boldsymbol{\Sigma}_{\boldsymbol{\Theta}_i}^{(s)} + \mathrm{E}(\tau_i | \Delta \boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}) \\
&\quad \times (\boldsymbol{\mu}_i^{(s)} - \Delta \boldsymbol{\Sigma}^{-1}(t_{i,k}) \Delta \boldsymbol{Y}_{i,k}) (\boldsymbol{\mu}_i^{(s)} - \Delta \boldsymbol{\Sigma}^{-1}(t_{i,k}) \Delta \boldsymbol{Y}_{i,k})' \Big].
\end{aligned}
$$

$$
-2 \ln \Gamma(\nu/2) + \nu \ln(\nu/2) + \frac{\nu}{n} \sum_{i=1}^{n} \left[\mathrm{E}\left(\ln \tau_i | \boldsymbol{\Delta Y}_i, \boldsymbol{\Psi}^{(s)}\right) - \mathrm{E}\left(\tau_i | \boldsymbol{\Delta Y}_i, \boldsymbol{\Psi}^{(s)}\right) \right].
$$

EM algorithm



Algorithm 2: Implementation of the proposed EM algorithm.

```
Input: $\mathbb{Y}, \boldsymbol{\Psi}^{(0)}, \epsilon$;
Output: $\hat{\boldsymbol{\Psi}} = \{\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\Sigma}}_\delta, \hat{\boldsymbol{\Sigma}}_0, \hat{\nu}\}$.
while $||\, \boldsymbol{\Psi}^{(s+1)} - \boldsymbol{\Psi}^{(s)} \,|| \geq \epsilon$ do
    E-step:
    Compute $\mathrm{E}[\tau_i|\Delta\boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}]$ and $\mathrm{E}\left[\ln \tau_i|\Delta\boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right]$ by (13);
    Compute $\mathrm{E}\left[\tau_i\ell_{i,0}|\Delta\boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right]$ and $\mathrm{E}\left[\tau_i\ell_{i,k}|\Delta\boldsymbol{Y}_i, \boldsymbol{\Psi}^{(s)}\right]$ by Theorem 2;
    M-step:
    Update $\boldsymbol{\Psi}^{(s+1)}$ by Theorem 3 and (15).
end
```

Interval estimation

```
Algorithm 3: Bootstrap algorithm procedure.
  Input: Point estimate $\hat{\boldsymbol{\Psi}}$ and $\hat{\boldsymbol{\gamma}}$.
  Output: $B$ resamples of the estimate $\left\{\hat{\boldsymbol{\Psi}}_1^*, \ldots, \hat{\boldsymbol{\Psi}}_B^*\right\}$.
for $b = 1$ to $B$ do
    for $i = 1$ to $n$ do
        Generate $\tilde{\tau}_i$ from $\mathcal{G}\left(\hat{v}/2, 2/\hat{v}\right)$;
        Generate $\tilde{\boldsymbol{\Theta}}$ from $\mathcal{N}_p(\hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\Sigma}}_0/\tilde{\boldsymbol{\tau}})$;
        for $k = 1$ to $m_i$ do
            Given $\tilde{\boldsymbol{\Theta}}_i$ and $\tilde{\tau}_i$, generate $\Delta\tilde{\boldsymbol{Y}}_{i,k}$ from $\mathcal{N}_p\left(\Delta\boldsymbol{\Sigma}(t_{i,k})\tilde{\boldsymbol{\Theta}}_i, \frac{\boldsymbol{\Omega}_\delta}{\tilde{\tau}_i}\Delta\boldsymbol{\Sigma}(t_{i,k})\right)$;
        end
    end
    Obtain the bootstrapped degradation data $\tilde{\mathbb{Y}}$;
    Obtain $\hat{\boldsymbol{\Psi}}_b^*$ based on $\tilde{\mathbb{Y}}$ using the proposed EM algorithm.
end
```

Interval estimation

Interval estimation

After obtaining the B bootstrap estimates $\left\{\hat{\boldsymbol{\Psi}}_1^*, \ldots, \hat{\boldsymbol{\Psi}}_B^*\right\}$, we can proceed to construct an approximate $100(1-\alpha)\%$ bootstrap confidence interval for a function of the parameters $h(\boldsymbol{\Psi})$. The interval estimation is constructed as follows:

$$\left[h\left(\hat{\boldsymbol{\Psi}}^*\right)_{(\alpha B/2)}, h\left(\hat{\boldsymbol{\Psi}}^*\right)_{((1-\alpha/2)B)}\right],$$

where $h\left(\hat{\boldsymbol{\Psi}}^*\right)_{(b)}$ is the b-th order statistic among $\left\{h\left(\hat{\boldsymbol{\Psi}}^*\right)_1, \ldots, h\left(\hat{\boldsymbol{\Psi}}^*\right)_B\right\}$

Outline

Experimental setup

1. Set the degrees of freedom to $\nu = 5$.
2. Assume periodic measurements at $t = 5, 10, ..., 5m$.
3. $n = 10, 20, 30,$ and $m = 10, 20, 30.$

Table 1: Four combinations of p and $\Lambda(t)$, along with their corresponding parameter setting.

Scen.	$\Lambda(t)$	p	η_1	η_2	η_3	δ_1	δ_2	δ_3	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{13}	σ_{23}	γ_1	γ_2	γ_3
I	t	2	11	12	-	0.5	1.5	-	0.5	1	-	0.75	-	-	-	-	-
II		3	11	12	13	0.4	0.6	0.8	1	2	3	0.75	-1.0	1.2	-	-	-
III	t^γ	2	11	12	-	0.5	1.5	-	0.5	1	-	0.75	-	-	0.5	1.5	-
IV		3	11	12	13	0.4	0.6	0.8	1	2	3	0.75	-1.0	1.2	0.8	1	1.2

Performance of the inference method

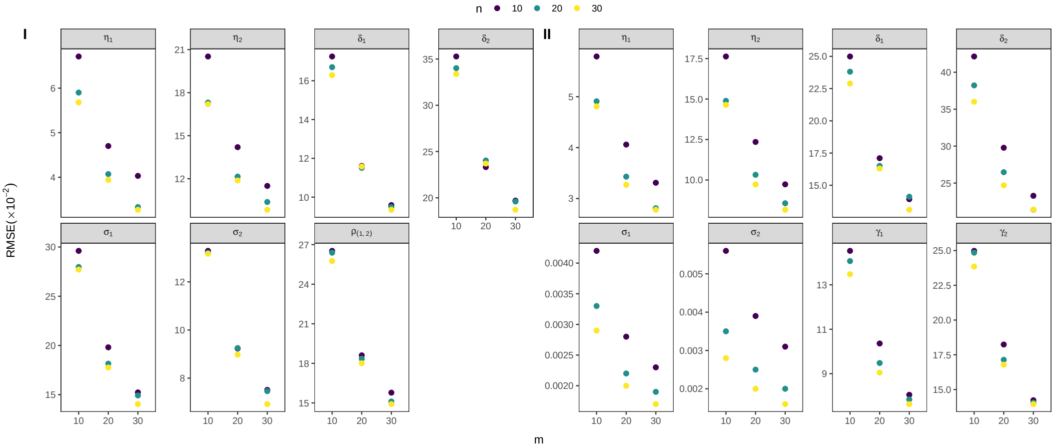


Figure 4: RMSE $(\times 10^{-2})$ for parameter estimators in scenarios I and III.

Performance of the inference method

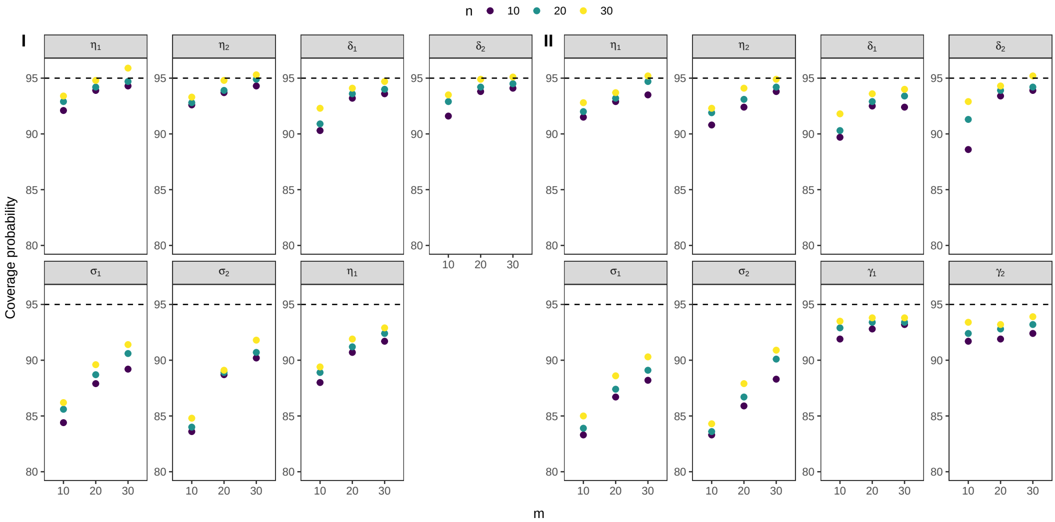


Figure 5: CP (%) for parameter estimators in scenarios I and III.

Effect of model misspecification

- **Multivariate Wiener process models** $(\nu \to \infty)$.
- Calculate the mean time to failure of the system, $\mathrm{MTTF} = E(T) = \int_0^\infty R_{T_\omega}(t)\mathrm{d}t$.

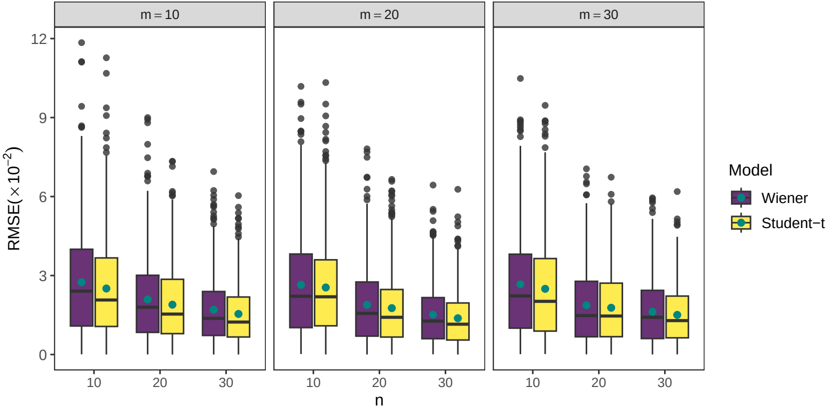


Figure 6: RMSE $(\times 10^{-2})$ for MTTF estimators under various sample sizes in scenario IV (green points mean the average RMSEs).

Outline

PMB degradation data

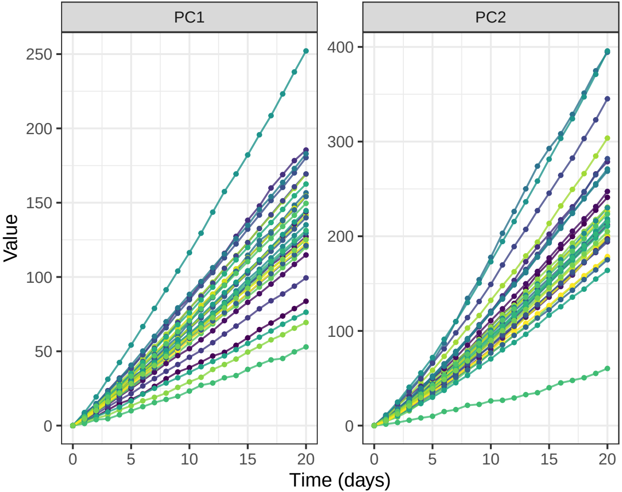


Figure 7: Degradation paths for the PMB data.

Table 2: Parameter point estimation and 90% CI for the PMB data, where M_l: linear form $\Lambda(t) = t$; M_p: power form $\Lambda(t) = t^\gamma$.

Model	M_l	M_p	M_l^W	M_p^W
η_1	6.412 (5.913, 6.829)	4.885 (4.483, 5.317)	6.528 (6.026, 6.986)	4.936 (4.582, 5.331)
η_2	10.727 (10.018, 11.313)	5.877 (5.413, 6.367)	10.822 (9.893, 11.441)	5.938 (5.424, 6.529)
δ_1	0.814 (0.702, 0.95)	0.382 (0.34, 0.439)	0.777 (0.741, 0.813)	0.449 (0.426, 0.48)
δ_2	2.102 (1.802, 2.475)	0.573 (0.511, 0.655)	2.054 (1.929, 2.154)	0.673 (0.636, 0.705)
σ_{11}	2.563 (1.399, 4.321)	1.067 (0.638, 1.757)	2.317 (1.481, 3.370)	1.577 (1.026, 2.340)
σ_{22}	6.213 (3.376, 9.083)	1.417 (0.874, 2.319)	5.227 (3.379, 7.460)	2.230 (1.441, 3.212)
σ_{12}	2.640 (1.387, 4.753)	0.835 (0.413, 1.459)	2.461 (1.282, 3.660)	1.344 (0.839, 2.300)
ν	1.777 (1.597, 1.946)	4.759 (3.185, 7.798)	-	-
γ_1	-	1.107 (1.094, 1.121)	-	1.106 (1.098, 1.115)
γ_2	-	1.212 (1.195, 1.227)	-	1.213 (1.201, 1.224)
AIC	3776.361	**1619.749**	4022.552	1794.181

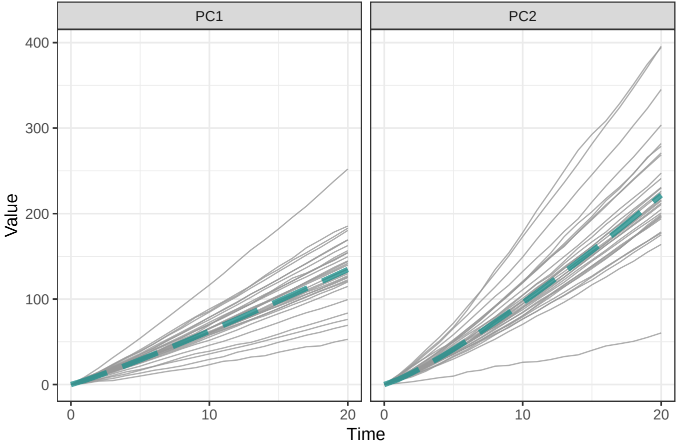


Figure 8: Estimation of average degradation path fitting results for the PMB data using model M_p.

System reliability analysis

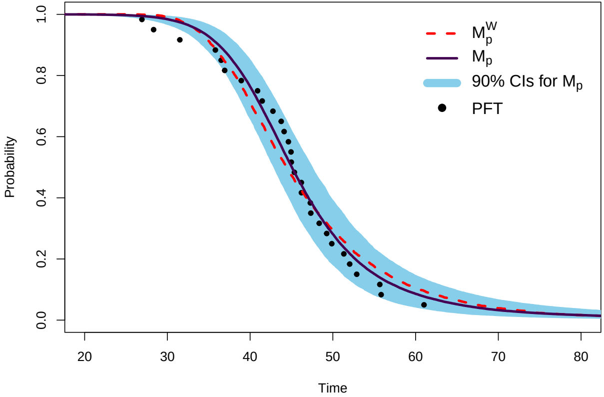


Figure 9: The estimated reliability of the PMB data.

Anderson-Darling test: p-values for M_p are 0.559, and for M_p^W are 0.289.

Fatigue crack-size data

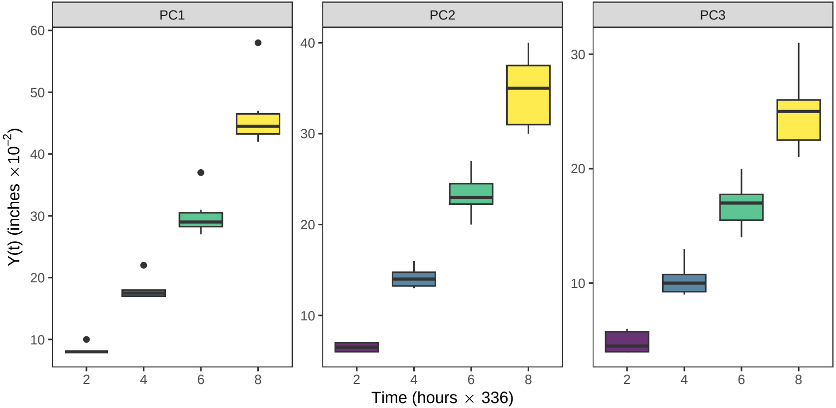


Figure 10: Degradation paths of the FCS data.

Table 3: Parameter point estimation and 90% CI for the FCS data.

Model	Parameters					
M_e	η_1	η_2	η_3	δ_1	δ_2	δ_3
	30.179	51.021	37.896	1.927	3.026	2.682
	(19.250, 50.228)	(23.098, 210.201)	(14.661, 609.305)	(1.512, 2.848)	(2.131, 7.224)	(1.761, 12.912)
	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{13}	σ_{23}
	12.066	26.594	23.540	17.911	16.852	25.019
	(1.377, 28.745)	(1.924, 66.389)	(2.111, 50.674)	(1.775, 35.023)	(2.046, 34.467)	(2.329, 55.109)
	γ_1	γ_2	γ_3	ν	AIC	
	0.117	0.065	0.062	27.440	**312.625**	
	(0.080, 0.151)	(0.018, 0.113)	(0.005, 0.120)	(8.338, 62.193)		
M_e^W	η_1	η_2	η_3	δ_1	δ_2	δ_3
	30.547	51.457	38.425	1.982	3.091	2.710
	(19.163, 47.492)	(25.477, 81.464)	(17.174, 60.959)	(1.57, 3.509)	(2.242, 21.648)	(1.89, 48.371)
	σ_{11}	σ_{22}	σ_{33}	σ_{12}	σ_{13}	σ_{23}
	14.574	27.590	24.981	20.036	19.072	26.246
	(1.195, 35.722)	(0.940, 95.973)	(0.955, 67.324)	(1.116, 45.057)	(1.281, 41.863)	(0.955, 77.997)
	γ_1	γ_2	γ_3	ν	AIC	
	0.117	0.065	0.062	-	328.1935	
	(0.082, 0.146)	(0.023, 0.110)	(0.005, 0.119)			

System reliability analysis

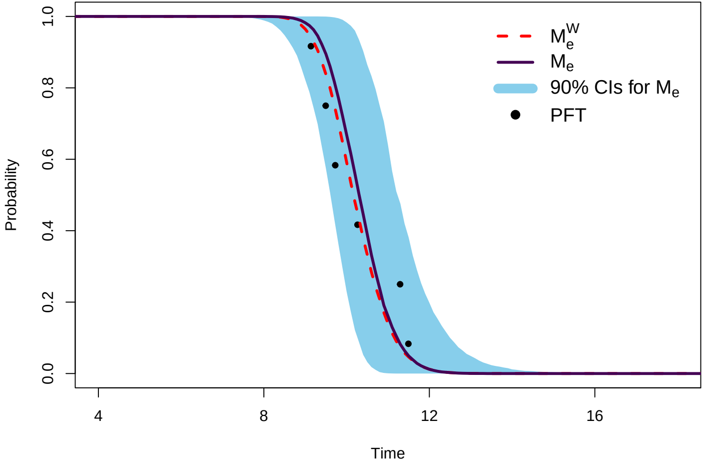


Figure 11: The estimated reliability of the FCS data.

Outline

Conclusion

1. Introduce a novel class of tail-weighted multivariate degradation models, accounting for both within-unit variability and dependencies among PCs while allowing flexible **tuning of the tail heaviness through the parameter of degree of freedom**.
2. Derive system reliability and provide an efficient MC method for reliability assessment.
3. Develop an innovative two-stage parameter estimation method, integrating NLS and EM methods, supplemented by bootstrap for constructing parameter interval estimates.
4. Comprehensive simulation studies are conducted to validate the effectiveness of our inference methods.
5. Demonstrate the effectiveness of our proposed methodology through case studies.

Thanks!