Reparameterized Inverse Gaussian Process and Its Applications

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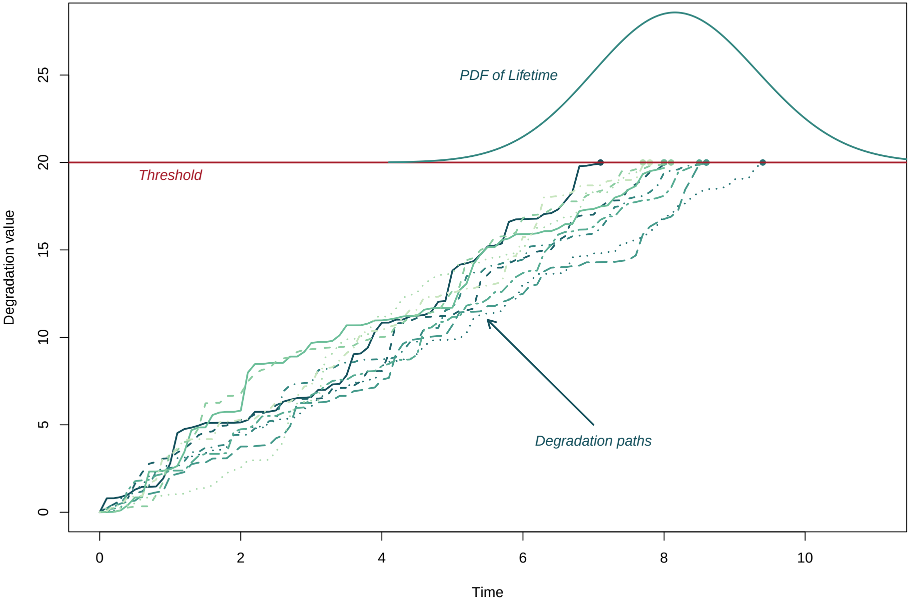
Outline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Degradation paths
Degradation value
Time
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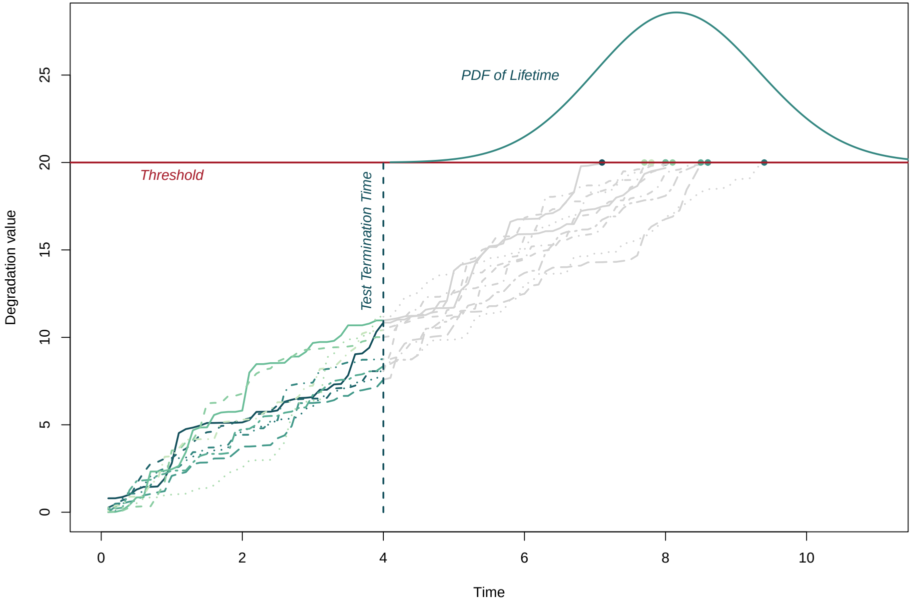
PDF of Lifetime
Threshold
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Degradation Models

- General path model.
- Stochastic process: Wiener, gamma, inverse Gaussian (IG), variance gamma, Ornstein–Uhlenbeck, etc.
- Review papers: Si et al. (2011), Ye and Xie (2015), Zhang et al. (2018).

Reparameterized IG (rIG) distribution

Connection to IG distribution

The rIG distribution $rIG(\delta, \gamma)$ relates to the traditional IG distribution $IG(a, b)$ as $a = \delta/\gamma$ and $b = \delta^2$.

Moment generating function (MGF)

$$M_Y(t) = E(e^{ty}) = e^{\delta\gamma\left(1-\sqrt{1-\frac{2t}{\gamma^2}}\right)}. \tag{1}$$

Additive property

If $Y_1 \sim rIG(\delta_1, \gamma), Y_2 \sim rIG(\delta_2, \gamma)$, then $Y_1 + Y_2 \sim rIG(\delta_1 + \delta_2, \gamma)$.

Probability density function (PDF)

If a random variable Y follows rIG distribution, then its PDF is

$$f_{rIG}(y|\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} y^{-3/2} e^{-\left(\delta^2 y^{-1} + \gamma^2 y\right)/2}, \; y > 0, \; \delta > 0, \; \gamma > 0. \quad (2)$$

Cumulative distribution function (CDF)

$$F_{rIG}(y|\delta,\gamma) = \Phi\left[\sqrt{y}\gamma - \frac{\delta}{\sqrt{y}}\right] + e^{2\delta\gamma}\Phi\left[-\sqrt{y}\gamma - \frac{\delta}{\sqrt{y}}\right], \quad (3)$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

rIG process

Definition of rIG process

rIG process $\{Z(t), t \geq 0\}$ satisfies the following properties:

(i) $Z(0) = 0$ with probability one;

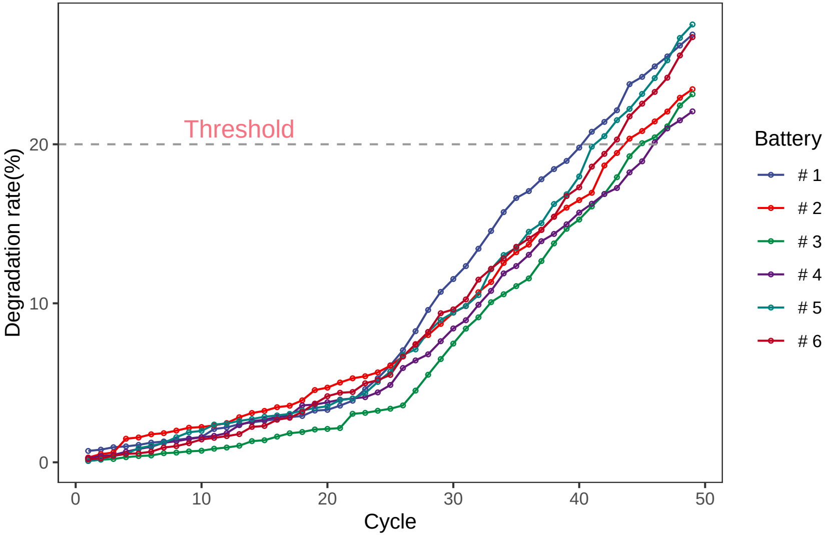
(ii) $Z(t)$ has independent increments. Specifically, $Z(t_2) - Z(t_1)$ and $Z(s_2) - Z(s_1)$ are independent for all $t_2 > t_1 \geq s_2 > s_1 \geq 0$;

(iii) For all $t > s \geq 0$, $Z(t) - Z(s)$ follows the rIG distribution $rIG(\delta(\Lambda(t) - \Lambda(s)), \gamma)$, where $\Lambda(t)$ is a monotone increasing function with $\Lambda(0) = 0$, δ and γ are unknown parameters.

- Denoted as $r\mathcal{IG}(\delta\Lambda(t), \gamma)$.
- The mean and variance of $\{Z(t), t \geq 0\}$, which are $\delta\Lambda(t)/\gamma$ and $\delta\Lambda(t)/\gamma^3$, respectively.

Outline

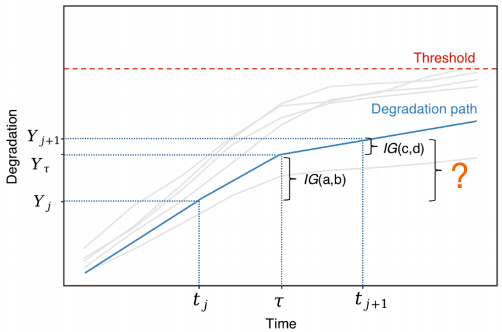
Two-stage degradation



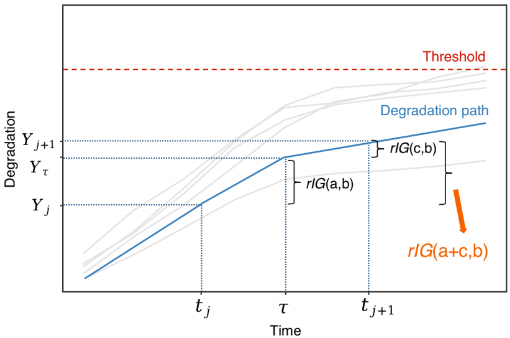
Related Literature

Two-phase degradation modeling

1. Wiener process: Wang et al. (2018a, 2018b), Zhang et al. (2019), Lin et al. (2021), Ma et al. (2023), etc.
2. Gamma process: Ling et al. (2019), Lin et al. (2021).
3. IG process: Duan and Wang (2017).
   - Limitations of Duan and Wang (2017):
     (i) Constraints on locations of change points;
     (ii) Insufficient considerations for deriving the lifetime distribution;
     (iii) Neglecting the uncertainty in estimation.



(a) IG process



(b) Re-parameterized IG process

Contributions

(i) A novel two-phase rIG degradation model with distinct change points and model parameters for each individual system;

(ii) Derive the distribution of failure time and RUL, and propose an adaptive replacement policy;

(iii) Employ bootstrap and Bayesian approach to generate interval estimates for the parameters.

Two-phase rIG degradation model

Two-phase rIG degradation model

Suppose a system's performance characteristic degrades in two distinct phases, separated by a single change point.

$$
\begin{aligned}
Y(t)|\tau \sim r\mathcal{IG}\left(m(t;\delta_1,\delta_2,\tau),\gamma\right),\ \tau \sim N\left(\mu_\tau,\sigma_\tau^2\right), \\
m(t;\delta_1,\delta_2,\tau) = \begin{cases} \delta_1 t, & t \leq \tau, \\ \delta_2\left(t-\tau\right)+\delta_1\tau, & t > \tau, \end{cases}
\end{aligned} \tag{4}
$$

where δ_1 and δ_2 are the drift parameters for $t \leq \tau$ and $t > \tau$, respectively.

Failure-time

Let $T = \inf\{t \mid Y(t) \geq \mathcal{D}\}$, and $Y(t) = \begin{cases} Y_1(t), & t \leq \tau, \\ Y_1(\tau) + Y_2(t-\tau), & t > \tau. \end{cases}$

Conditional reliability function of T

- $0 \leq t \leq \tau$

$$\bar{F}_1(t \mid \tau) = P(T > t \mid \tau \geq t) = P(Y_1(t) < \mathcal{D} \mid \tau \geq t) = F_{r\mathcal{IG}}(\mathcal{D}|\delta_1 t, \gamma). \tag{5}$$

- $t > \tau$

$$\begin{aligned} \bar{F}_2(t \mid \tau) =& P\left(Y(t) < \mathcal{D} \mid \tau < t\right) = P\left(Y_1(\tau) + Y_2(t-\tau) < \mathcal{D} \mid \tau < t\right) \\ =& \int_0^{\mathcal{D}} F_{r\mathcal{IG}}(\mathcal{D} - y_\tau | \delta_2(t-\tau), \gamma) f_1(y_\tau \mid \tau) \mathrm{d}y_\tau, \end{aligned} \tag{6}$$

where y_τ represents the degradation value at τ, and $f_1(y_\tau \mid \tau)$ is the PDF of y_τ.

Failure-time

Unconditional reliability function of T

$$\begin{aligned} R(t) &= P\left(Y(t) < \mathcal{D}, \tau \geq t\right) + P\left(Y(t) < \mathcal{D}, 0 < \tau < t\right) \\ &= \bar{F}_1\left(t \mid \tau\right) \bar{G}_\tau(t) + \int_0^t g_\tau(\tau|\mu_\tau, \sigma_\tau^2) \bar{F}_2\left(t \mid \tau\right) d\tau, \end{aligned} \tag{7}$$

where $\bar{G}_\tau(t)$ is the survival function of random variable τ.

Mean time to failure (MTTF)

$$\text{MTTF} = E(T) = \int_0^\infty R(t) dt. \tag{8}$$

Remaining useful life (RUL)

Let $S_t = \inf\left\{x; Y\left(t+x\right) \geq \mathcal{D} \mid Y(t) < \mathcal{D}\right\}$.

Conditional reliability function of S_t

(i) When $x+t \leq \tau$:

$$\bar{F}_{S_t,1}(x \mid \tau) = F_{r\mathcal{IG}}(\mathcal{D} - Y(t)|\delta_1 x, \gamma). \tag{9}$$

(ii) When $t < \tau < x+t$:

$$\begin{aligned}\bar{F}_{S_t,2}(x \mid \tau) &= P(Y(t+x) < \mathcal{D} \mid Y(t) \leq \mathcal{D}) \\ &= \int_0^{\mathcal{D}} F_{r\mathcal{IG}}(\mathcal{D} - y_\tau|\delta_2(t+x-\tau), \gamma) f_1(y_\tau \mid \tau)\mathrm{d}y_\tau.\end{aligned} \tag{10}$$

(iii) When $\tau \leq t$:

$$\bar{F}_{S_t,3}(x \mid \tau) = F_{r\mathcal{IG}}(\mathcal{D} - Y(t)|\delta_2 x, \gamma). \tag{11}$$

RUL

Unconditional reliability function of S_t

$$
\begin{aligned}
R_{S_t}(x) =& P(Y(t+x) < \mathcal{D}, t < x+t \leq \tau) \\
& + P(Y(t+x) < \mathcal{D}, t \leq \tau < x+t) + P(Y(t+x) < \mathcal{D}, t > \tau) \\
=& \bar{F}_{S_t,1}(x \mid \tau)\,\bar{G}_\tau(x+t) + \int_t^{x+t} g_\tau(\tau|\mu_\tau, \sigma_\tau^2)\bar{F}_{S_t,2}(x \mid \tau)\,d\tau \\
& + \int_0^t g_\tau(\tau)\bar{F}_{S_t,3}(x \mid \tau)\,d\tau.
\end{aligned}
\tag{12}
$$

Mean of RUL at time t

$$
\text{MRL} = E(S_t) = \int_0^\infty R_{S_t}(x)dx. \tag{13}
$$

Data

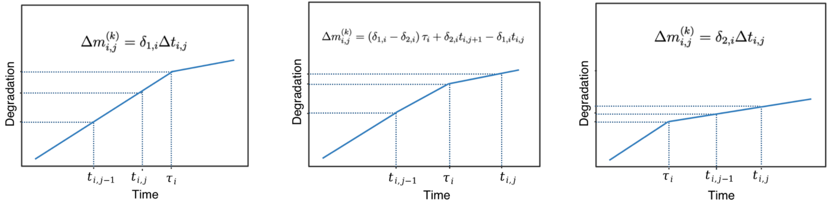
- I systems under inspection in a degradation test.
- Deterioration pattern follows the two-phase rIG degradation model.
- $Y_{i,j}$ is the observed degradation value at the measurement time $t_{i,j}$, $i = 1 \ldots, I,\ j = 1, \ldots, n_i$, and $0 < t_{i,1} < \ldots < t_{i,n_i}$.
- Let $\Delta y_{i,j} = Y_{i,j} - Y_{i,j-1}$, $Y_{i,0} = 0$.
- Denote $\boldsymbol{\Delta Y}_i = (\Delta y_{i,1}, \ldots, \Delta y_{i,n_i})^\top$, $\boldsymbol{\Delta Y} = \left(\boldsymbol{\Delta Y}_1^\top, \cdots, \boldsymbol{\Delta Y}_I^\top\right)^\top$.

Conditional PDF of $\Delta y_{i,j}$

$$\Delta y_{i,j} \sim rIG\left(\Delta m_{i,j}^{(k)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right), \gamma\right),$$

$$\Delta m_{i,j}^{(k)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right) = \begin{cases} \delta_{1,i}\Delta t_{i,j} & k=1, \\ \left(\delta_{1,i}-\delta_{2,i}\right)\tau_i + \delta_{2,i}t_{i,j} - \delta_{1,i}t_{i,j-1}, & k=2, \\ \delta_{2,i}\Delta t_{i,j}, & k=3, \end{cases}$$

$\Delta t_{i,j} = t_{i,j} - t_{i,j-1}$ and $t_{i,0} = 0$, $i = 1\ldots, I$, $j = 1, \ldots, n_i$.



(a) $\tau_i \geq t_{i,j}$ (b) $t_{i,j-1} \leq \tau_i < t_{i,j}$ (c) $\tau_i < t_{i,j-1}$

Figure 1: Three scenarios for change points and inspection time.

Conditional PDF of $\Delta y_{i,j}$

Let $\lambda_{i,j}^{(1)} = \mathcal{I}\left(\tau_i \geq t_{i,j}\right), \lambda_{i,j}^{(2)} = \mathcal{I}\left(t_{i,j-1} \leq \tau_i < t_{i,j}\right), \lambda_{i,j}^{(3)} = \mathcal{I}\left(\tau_i < t_{i,j-1}\right)$.

$$\Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right) = \Delta m_{i,j}^{(1)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)^{\lambda_{i,j}^{(1)}} \times \Delta m_{i,j}^{(2)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)^{\lambda_{i,j}^{(2)}} \times \Delta m_{i,j}^{(3)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)^{\lambda_{i,j}^{(3)}}.$$

$$\begin{aligned} f_{i,j}\left(\Delta y_{i,j} \mid \delta_{1,i}, \delta_{2,i}, \tau_i, \gamma\right) = & \frac{\Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)}{\sqrt{2\pi}} \exp\left\{\gamma \Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)\right\} \Delta y_{i,j}^{-3/2} \\ & \times \exp\left\{-\frac{\left[\Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)\right]^2 \Delta y_{i,j}^{-1} + \gamma^2 \Delta y_{i,j}}{2}\right\}. \end{aligned}$$

Likelihood

- Let $\boldsymbol{\delta}_1 = (\delta_{1,1}, \ldots, \delta_{1,I})^\top$, $\boldsymbol{\delta}_2 = (\delta_{2,1}, \ldots, \delta_{2,I})^\top$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_I)^\top$.
- Denote $\boldsymbol{\eta} = \left(\boldsymbol{\delta}_1^\top, \boldsymbol{\delta}_2^\top, \gamma\right)^\top$, $\boldsymbol{\theta}_\tau = \left(\mu_\tau, \sigma_\tau^2\right)^\top$ and $\boldsymbol{\vartheta} = \left(\boldsymbol{\theta}_\tau^\top, \boldsymbol{\eta}^\top\right)^\top$.
- Given the observed data $\boldsymbol{\Delta Y}$, the likelihood function is

$$L_{obs}(\boldsymbol{\Delta Y}|\boldsymbol{\vartheta}) = \prod_{i=1}^{I} \int_{-\infty}^{\infty} \prod_{j=1}^{n_i} f_{i,j}\left(\Delta y_{i,j} \mid \delta_{1,i}, \delta_{2,i}, \tau_i, \gamma\right) g_\tau(\tau_i|\boldsymbol{\theta}_\tau)\mathrm{d}\tau_i. \tag{14}$$

Remark: Obtain a closed-form solution for the ML estimates of $\boldsymbol{\vartheta}$ is not feasible.

EM Algorithm

Log-likelihood function for the complete data

$$l_c(\boldsymbol{\Delta Y}, \boldsymbol{\tau} | \boldsymbol{\vartheta}) = \sum_{i=1}^{I} l_i\left(\boldsymbol{\theta}_\tau\right) + \sum_{i=1}^{I}\sum_{j=1}^{n_i} l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}), \tag{15}$$

$$l_i\left(\boldsymbol{\theta}_\tau\right) = \log g_\tau\left(\tau_i \mid \boldsymbol{\theta}_\tau\right) = -\log\sqrt{2\pi}\sigma_\tau - \frac{\left(\tau_i - \mu_\tau\right)^2}{2\sigma_\tau^2},$$

$$\begin{aligned} l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}) &= \log f_{i,j}\left(\Delta y_{i,j} \mid \boldsymbol{\eta}, \boldsymbol{\tau}\right) \\ &= -\log\sqrt{2\pi} + \log\Delta m_{i,j} + \gamma\Delta m_{i,j} - \frac{3}{2}\log\Delta y_{i,j} - \frac{\Delta m_{i,j}^2}{2\Delta y_{i,j}} - \frac{\gamma^2\Delta y_{i,j}}{2}, \end{aligned}$$

and $\Delta m_{i,j} = \Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)$.

EM Algorithm

- E-step:

$$
\begin{aligned}
\boldsymbol{Q}_{(s)}(\boldsymbol{\vartheta}) &= E_{\boldsymbol{\vartheta}_{(s)}}\left[l_c(\boldsymbol{\Delta Y}, \boldsymbol{\tau}|\boldsymbol{\vartheta})\right] \\
&= \sum_{i=1}^{I} E_{\boldsymbol{\vartheta}_{(s)}}\left[l_i\left(\boldsymbol{\theta}_\tau\right) \mid \boldsymbol{\Delta Y}\right] + \sum_{i=1}^{I}\sum_{j=1}^{n_i} E_{\boldsymbol{\vartheta}_{(s)}}\left[l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}) \mid \boldsymbol{\Delta Y}\right],
\end{aligned} \tag{16}
$$

- M-step:

$$
\boldsymbol{\vartheta}_{(s+1)} = \arg\max \boldsymbol{Q}_{(s)}(\boldsymbol{\vartheta}). \tag{17}
$$

EM Algorithm

- **Step 1**. Initialize the parameters $\boldsymbol{\vartheta}$ to some random values $\boldsymbol{\vartheta}_{(\mathbf{0})}$, and setting the tolerance error ϵ.
- **Step 2**. Calculate $E_{\boldsymbol{\vartheta}_{(s)}}\left[l_i\left(\boldsymbol{\theta}_\tau\right) \mid \boldsymbol{\Delta y}\right]$ and $E_{\boldsymbol{\vartheta}_{(s)}}\left[l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}) \mid \boldsymbol{\Delta y}\right]$ based on the solution of the s-th iteration $\boldsymbol{\vartheta}_{(s)}$.
- **Step 3**. Calculate the solution of the $(s+1)$-th iteration $\boldsymbol{\vartheta}_{(s+1)}$ by (17).
- **Step 4**. Repeat Steps 2 and 3 until $\left|\boldsymbol{\vartheta}_{(s+1)} - \boldsymbol{\vartheta}_{(s)}\right| < \epsilon$, where $|\cdot|$ is the Euclidean distance.
- **Step 5**. The MLE of $\boldsymbol{\vartheta}$ can be obtained as $\hat{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}_{(s+1)}$.

Parametric bootstrap method

Algorithm 1: Parametric bootstrap algorithm.

Input: Point estimate $\hat{\boldsymbol{\vartheta}}$.

Output: $\mathcal{B}$ bootstrap estimates $\left\{\hat{\boldsymbol{\vartheta}}_1^*, \ldots, \hat{\boldsymbol{\vartheta}}_{\mathcal{B}}^*\right\}$.

for $b = 1$ **to** $\mathcal{B}$ **do**

  Generate $\boldsymbol{\tau}$ from $\mathcal{N}(\hat{\mu}_\tau, \hat{\sigma}_\tau^2)$;

  for $i = 1$ **to** I **do**

    for $j = 1$ **to** n_i **do**

      Generate degradation sample $\Delta\tilde{Y}_{i,j}$ from $rIG\left(\Delta m_{i,j}^{(k)}\left(\hat{\delta}_{1,i}, \hat{\delta}_{2,i}, \hat{\tau}_i\right), \hat{\gamma}\right), k = 1, 2, 3.$

    end

  end

  Obtain $\hat{\boldsymbol{\vartheta}}_b^*$ based on $\Delta\tilde{\boldsymbol{Y}}$ using the proposed EM algorithm.

end

Parametric bootstrap method

After acquiring the bootstrap estimates $\left\{\hat{\boldsymbol{\vartheta}}_1^*, \ldots, \hat{\boldsymbol{\vartheta}}_{\mathcal{B}}^*\right\}$, an approximate $100(1-\alpha)\%$ bootstrap confidence interval for a function of the parameters $h(\boldsymbol{\vartheta})$ is:

$$\left[h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{(\alpha\mathcal{B}/2)}, h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{((1-\alpha/2)\mathcal{B})}\right],$$

where $h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{(b)}$ denotes the b-th statistic among $\left\{h\left(\hat{\boldsymbol{\vartheta}}^*\right)_1, \ldots, h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{\mathcal{B}}\right\}$.

Bayesian analysis

$$Y_i(t|\tau_i) \sim r\mathcal{IG}\left(m(t;\delta_{1,i},\delta_{2,i},\tau_i),\gamma\right),\ \tau_i \sim N\left(\mu_\tau,\sigma_\tau^2\right),\ i=1,\ldots,I,$$

$$m(t;\delta_{1,i},\delta_{2,i},\tau_i) = \begin{cases} \delta_{1,i}t, & t \le \tau_i, \\ \delta_{2,i}\left(t-\tau_i\right)+\delta_{1,i}\tau_i, & t > \tau_i, \end{cases}$$

$$\left(\mu_\tau,\sigma_\tau^2\right) \sim NIGa\left(\beta_\tau,\eta_\tau,\upsilon_\tau,\xi_\tau\right), \gamma \sim N(\omega,\kappa^2),$$

$$\delta_{1,i} \sim N\left(\mu_1,\sigma_1^2\right), \delta_{2,i} \sim N\left(\mu_2,\sigma_2^2\right),$$

$$\left(\mu_1,\sigma_1^2\right) \sim NIGa\left(\beta_1,\eta_1,\upsilon_1,\xi_1\right), \left(\mu_2,\sigma_2^2\right) \sim NIGa\left(\beta_2,\eta_2,\upsilon_2,\xi_2\right),$$

where $NIGa(\cdot)$ denotes the normal-inverse gamma distribution.

Joint posterior distribution of $\boldsymbol{\theta}$

- Let $\boldsymbol{\theta} = \left(\boldsymbol{\vartheta}, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2\right)^\top$ be the parameter vector.
- According to Bayes' theorem, the joint posterior distribution of $\boldsymbol{\theta}$ can be derived as

$$\begin{aligned}\pi(\boldsymbol{\theta} \mid \boldsymbol{\Delta Y}) \propto \pi\left(\mu_\tau, \sigma_\tau^2\right) \pi\left(\mu_1, \sigma_1^2\right) \pi\left(\mu_2, \sigma_2^2\right) \pi\left(\gamma \mid \omega, \kappa\right) \pi\left(\tau \mid \mu_\tau, \sigma_\tau^2\right) \\ \times \pi\left(\boldsymbol{\delta}_1 \mid \mu_1, \sigma_1^2\right) \pi\left(\boldsymbol{\delta}_2 \mid \mu_1, \sigma_1^2\right) f_{\Delta Y}\left(\boldsymbol{\Delta Y} \mid \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\tau}, \gamma\right).\end{aligned} \tag{18}$$

- Employ the **Gibbs sampling algorithm** to generate posterior samples of the parameters, thereby facilitating Bayesian inference.

Lithium-ion batteries

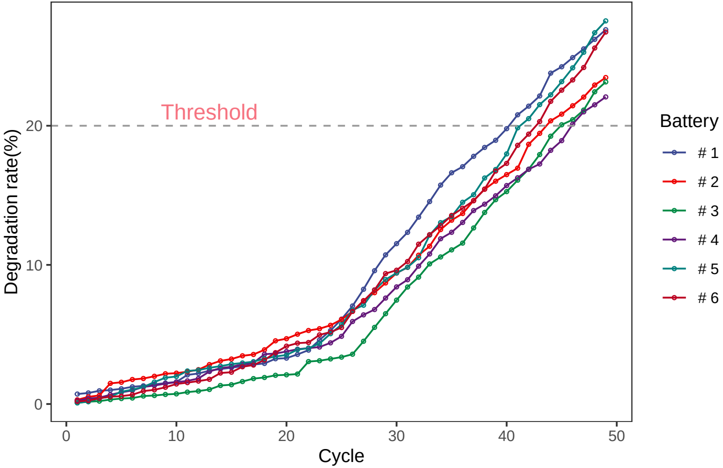


Figure 2: Capacity degradation data of 6 lithium batteries.

Parameter Estimation by two-phase rIG Model

Table 1: Parameter estimation based on the proposed model.

		HB			ML				HB			ML	
		β_1	β_2	τ	β_1	β_2			β_1	β_2	τ	β_1	β_2
	2.5%	0.422	2.198	22.257	0.497	2.511		2.5%	0.467	1.993	24.151	0.561	2.120
# 1	Mean	0.532	2.516	23.187	0.510	2.632	# 4	Mean	0.583	2.291	25.008	0.576	2.221
	97.5%	0.645	2.851	24.664	0.518	2.713		97.5%	0.703	2.595	26.060	0.587	2.288
	2.5%	0.523	2.013	24.365	0.638	2.113		2.5%	0.495	2.162	23.184	0.624	2.382
# 2	Mean	0.653	2.312	25.336	0.658	2.215	# 5	Mean	0.621	2.472	24.003	0.642	2.496
	97.5%	0.785	2.615	26.557	0.670	2.282		97.5%	0.752	2.809	25.370	0.654	2.572
	2.5%	0.336	2.161	26.316	0.405	2.412		2.5%	0.464	2.130	24.722	0.559	2.324
# 3	Mean	0.428	2.487	26.761	0.414	2.531	# 6	Mean	0.577	2.443	25.583	0.574	2.440
	97.5%	0.518	2.831	27.381	0.420	2.610		97.5%	0.697	2.769	26.306	0.585	2.517

Table 2: RMSE and RB results for different models.

Model	Training(30)		Prediciton (19)		Overall	
	RMSE	RB	RMSE	RB	RMSE	RB
Proposed	0.448	0.248	1.538	0.060	1.020	0.175
Linear	3.476	1.442	3.685	0.156	3.558	0.943
Power	2.057	0.568	2.475	0.113	2.229	0.391
Exp	0.908	0.313	1.611	0.065	1.230	0.217
Duan	0.434	0.239	1.976	0.075	1.276	0.175

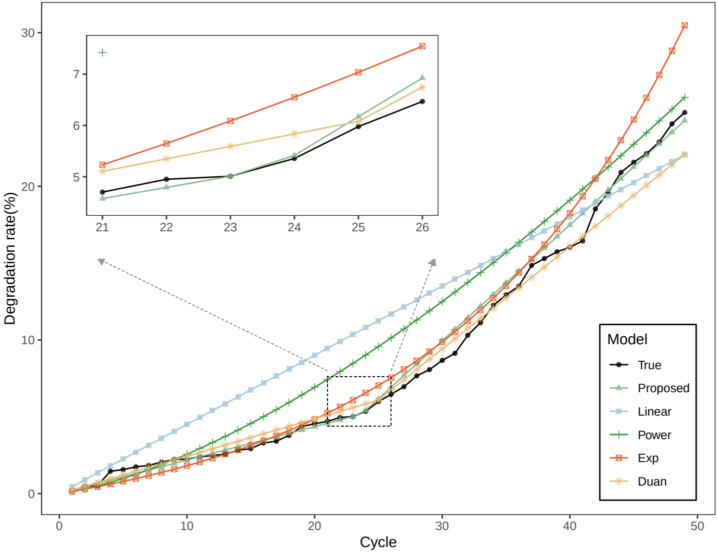


Figure 3: Degradation path training and prediction results for battery #2 using different methods, with a zoomed-in view of the potential change point locations.

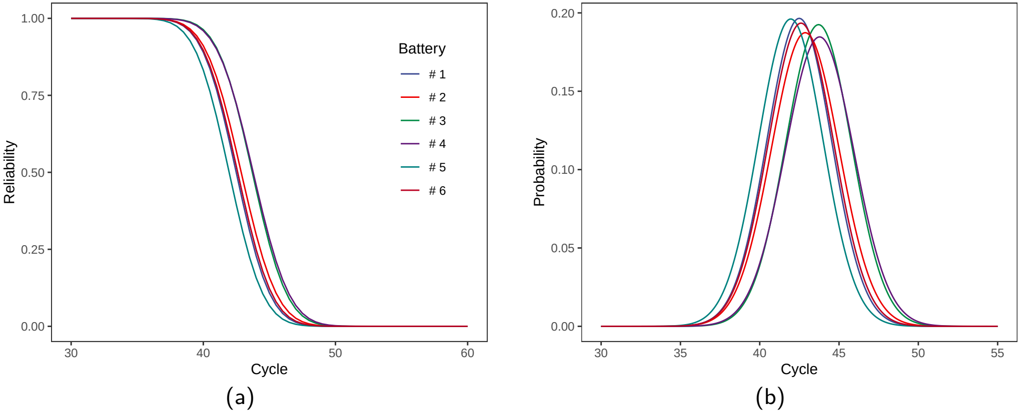


Figure 4: Reliability and density functions of failure time based on HB method.

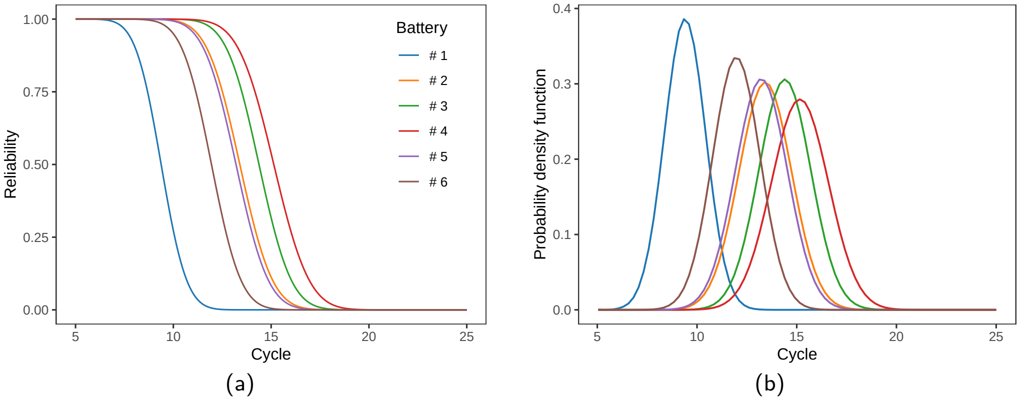
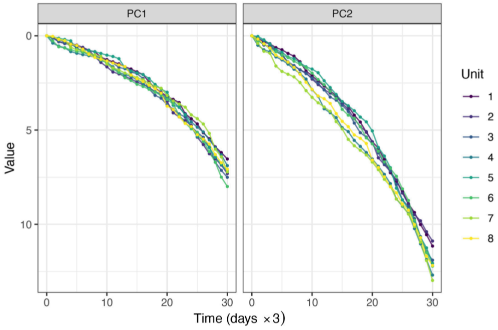


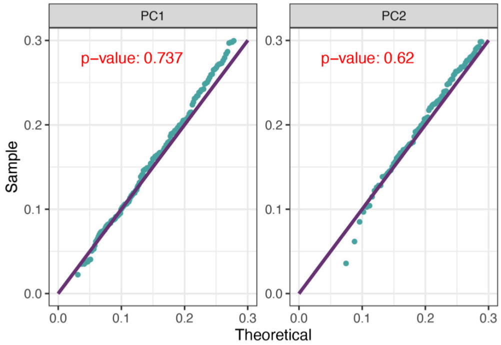
Figure 5: Reliability and density functions of RUL based on HB method.

Outline

Permanent magnet brake (PMB) data



(a) Degradation paths



(b) Q-Q plots using IG distribution

Figure 6: Summary of PMB data for two PCs: degradation paths and Q-Q plots.

PMB data

Unit	1	2	3	4	5	6	7	8
Correlation	0.819	0.749	0.806	0.840	0.779	0.749	0.765	0.800

Figure 7: Pearson correlation coefficients of two PCs across various units.

- **Objective**: establish a multivariate IG process model incorporating common effects.

Related literature

Multivariate degradation modeling

- Copula-based method
- Multivariate distribution-based method
- Common-effect-based method
  a) Frailty model-based method
  b) Stochastic process summation method

Challenges

1. **Copula-based method**: Faces difficulties in selecting appropriate copulas and providing clear physical interpretations.
2. **Multivariate distribution-based method**: Mostly limited to bivariate cases, with challenges in extending to multivariate distributions.
3. **Frailty model-based method**: The use of a single frailty factor limits the model's generality.

Advantages of stochastic process summation

- Model parameters increase linearly with dimensionality, simplifying high-dimensional degradation modeling.

Contributions

(i) Construct a multivariate rIG process using the common-effect method and analyze its properties and system lifetime distribution.

(ii) Apply Gauss-Legendre (GL) quadrature for approximating the complex integral in the lifetime distribution.

(iii) Use the EM algorithm for parameter estimation, with parametric bootstrap for confidence intervals.

Model definition

Degradation process of the k-th PC

$$Y_k(t) = X_k(t) + Z(t), k = 1, \ldots, K, \tag{19}$$

where $Z(t) \sim r\mathcal{IG}\left(\Lambda_0(t), \gamma\right)$ and $X_k(t) \sim r\mathcal{IG}\left(\Lambda_k(t), \gamma\right)$ are independent of each other, $\Lambda_k(t)$ and $\Lambda_0(t)$ are monotonically increasing functions of t.

Based on the additive property of the rIG distribution, $Y_k(t)$ is

$$Y_k(t) \sim rIG\left(\Lambda_k(t) + \Lambda_0(t), \gamma\right), k = 1, \ldots, K. \tag{20}$$

Proposition 1

The mean and variance of the degradation process $Y_k(t)$ are

$$\mathrm{E}\left[Y_k(t)\right] = \frac{\Lambda_0(t) + \Lambda_k(t)}{\gamma}, \quad \text{and} \quad \mathrm{Var}\left[Y_k(t)\right] = \frac{\Lambda_0(t) + \Lambda_k(t)}{\gamma^3}, \tag{21}$$

respectively. Meanwhile, the common effect $Z(t)$ introduces dependence among the multiple degradation processes

$$\mathrm{Cov}\left[Y_{k_1}(t_1), Y_{k_2}(t_2)\right] = \frac{\min\left(\Lambda_0(t_1), \Lambda_0(t_2)\right)}{\gamma^3}, k_1 \neq k_2. \tag{22}$$

At any given time t, Pearson correlation coefficient is

$$\rho\left[Y_{k_1}(t), Y_{k_2}(t)\right] = \frac{\Lambda_0(t)}{\sqrt{\left(\Lambda_0(t) + \Lambda_{k_1}(t)\right)\left(\Lambda_0(t) + \Lambda_{k_2}(t)\right)}}, k_1 \neq k_2. \tag{23}$$

Proposition 2: Joint PDF and CDF of $Y_1(t), \ldots, Y_K(t)$

$$f_{Y(t)}\left(y_1, \ldots, y_K\right) = \int_0^{\tilde{y}} f_{rIG}\left(z; \Lambda_0(t), \gamma\right) \prod_{k=1}^{K} f_{rIG}\left(y_k - z; \Lambda_k(t), \gamma\right) \mathrm{d}z,$$

where $\tilde{y} = \min\{y_1, \ldots, y_K\}$, where $y_1, \ldots, y_K$ are the observed dagradation values, $f_{rIG}(\cdot)$ is given by (2). The CDF is expressed as

$$F_{Y(t)}\left(y_1, \ldots, y_K\right) = \int_0^{\tilde{y}} f_{rIG}\left(z; \Lambda_0(t), \gamma\right) \prod_{k=1}^{K} F_{rIG}\left(y_k - z; \Lambda_k(t), \gamma\right) \mathrm{d}z,$$

where $F_{rIG}(\cdot)$ is given by (3).

System failure-time

Let $T_{\boldsymbol{\mathcal{D}}} = \inf\{t : Y_1(t) \geq \mathcal{D}_1$ or $\cdots$ or $Y_K(t) \geq \mathcal{D}_K\}$, where $\boldsymbol{\mathcal{D}} = (\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_K)'$ is a vector storing all PC failure thresholds.

Proposition 3: CDF of system failure time $T_{\boldsymbol{\mathcal{D}}}$

$$F_{T_{\boldsymbol{\mathcal{D}}}}(t \mid \boldsymbol{\Lambda}(t), \gamma, \boldsymbol{\mathcal{D}}) = \int_0^{\tilde{y}} \left[1 - \prod_{k=1}^{K} \left(F_{rIG}(\mathcal{D}_k - z; \Lambda_k(t), \gamma)\right)\right] f_{rIG}\left(z; \Lambda_0(t), \gamma\right) \mathrm{d}z,$$

where $\boldsymbol{\Lambda}(t) = (\Lambda_0(t), \ldots, \Lambda_K(t))'$, and $\tilde{y} = \min\{y_1, \ldots, y_K\}$.

Integral approximation

GL quadrature method

CDF of system failure-time can be approximated as

$$F_{T_{\boldsymbol{\mathcal{D}}}}(t \mid \boldsymbol{\Lambda}(t), \gamma, \boldsymbol{\mathcal{D}})$$
$$\approx \frac{\tilde{y}}{2} \sum_{q=1}^{l} w_q \left[1 - \prod_{k=1}^{K} \left(F_{rIG}(\mathcal{D}_k - \frac{\tilde{y}(u_q+1)}{2}; \Lambda_k(t), \gamma) \right) \right] f_{rIG} \left(\frac{\tilde{y}(u_q+1)}{2}; \Lambda_0(t), \gamma \right).$$

where l is a given order, u_q is the root of the Legendre polynomial and w_q is the corresponding weight.

Data

- n systems are tested in an experiment.
- The degradation of the K PCs in the i-th system are measured at m_i time points, denoted as $\boldsymbol{T}_i = (t_{i,1}, \ldots, t_{i,m_i})'$,
- Degradation values are $\boldsymbol{Y}_{i,k} = (Y_{i,k,1}, \ldots, Y_{i,k,m_i})'$ for $k = 1, \ldots, K$, $i = 1, \ldots, n$.
- The degradation increments of the k-th PC between $(t_{i,j-1}, t_{i,j}]$ as $\Delta Y_{i,k,j} \triangleq Y_{i,k,j} - Y_{i,k,j-1}$ for $j = 1, \ldots, m_i$.
- Set $t_{i,0} = 0$ and $Y_{i,k,0} = 0$.
- Denote $\Delta \boldsymbol{Y}_{i,:,j} = (\Delta Y_{i,1,j}, \ldots, \Delta Y_{i,K,j})'$.

Parameter

- $\Lambda_k(t) = \Lambda_k(t; \alpha_k, \beta_k)$ involves unknown parameters α_k and β_k, where $k = 0, \ldots, K$.
- Power-law form $\Lambda_k(t) = \beta_k t^{\alpha_k}$ and log-linear form $\Lambda_k(t) = \beta_k [\exp(\alpha_k t) - 1]$.
- For parameter nonidentifiability problem, we assume $\Lambda_0(t) = \Lambda_0(t; \alpha_0)$.
- Let $\mathbf{\Lambda}(t) = (\Lambda_0(t), \ldots, \Lambda_K(t))'$.
- Model parameters are $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \boldsymbol{\alpha}, \gamma\}$, with $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_K)'$ and $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_K)'$.

Likelihood

Given the observed data $\Delta \boldsymbol{Y}_{i,:,j}, i = 1, \ldots, n, j = 1, \ldots, m_i$,

Likelihood function

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \ln p\left(\Delta \boldsymbol{Y}_{i,:,j} \mid \boldsymbol{\theta}\right)$$

with

$$\begin{aligned} p\left(\Delta \boldsymbol{Y}_{i,:,j} \mid \boldsymbol{\theta}\right) = & \int_0^{\Delta \tilde{y}_{i,j}} f_{rIG}\left(\Delta z_{i,j}; \Delta\Lambda_0(t_{i,j}), \gamma\right) \\ & \times \prod_{k=1}^{K} f_{rIG}\left(\Delta y_{i,k,j} - \Delta z_{i,j}; \Delta\Lambda_k(t_{i,j}), \gamma\right) \mathrm{d}\Delta z_{i,j}, \end{aligned} \tag{24}$$

where $\Delta \tilde{y}_{i,j} = \min\{\Delta Y_{i,1,j}, \ldots, \Delta Y_{i,K,j}\}$.

EM algorithm

- Consider $Z_{i,j} = Z_i\left(t_{i,j}\right)$ for $i = 1, \ldots, n, j = 1, \ldots, m_i$ as the missing data;
- Define $\boldsymbol{Z}_i = (Z_{i,1}, \ldots, Z_{i,m_i})'$, $\Delta\Lambda_{i,k,j} = \Delta\Lambda_k(t_{i,j}), k = 0, \ldots, K$.
- $\Delta Y_{i,k,j} - \Delta Z_{i,j} \mid \boldsymbol{Z}_i \sim r\mathcal{IG}\left(\Delta\Lambda_{i,k,j}, \gamma\right)$, with $0 \leq \Delta Z_{i,j} \leq \Delta\tilde{y}_{i,j}$.
- Denote $\mathbb{Y} = \{\Delta\boldsymbol{Y}_{i,:,j}, i = 1, \ldots, n, j = 1, \ldots, m_i\}$, and $\mathbb{Z} = \{\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n\}$.

EM algorithm

Log-likelihood with the complete data

$$\ell(\boldsymbol{\theta} \mid \mathbb{Y}, \mathbb{Z}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left\{ \sum_{k=1}^{K} \ln p\left(\Delta Y_{i,k,j} - \Delta Z_{i,j} \mid \Delta Z_{i,j}\right) + \ln p\left(\Delta Z_{i,j}\right) \right\}, \tag{25}$$

$$\begin{aligned}\ln p\left(\Delta Y_{i,k,j} - \Delta Z_{i,j} \mid \Delta Z_{i,j}\right) = &-\frac{1}{2}\ln(2\pi) + \ln \Delta\Lambda_{i,k,j} - \frac{3}{2}\ln\left(\Delta Y_{i,k,j} - \Delta Z_{i,j}\right) \\ &+ \gamma \Delta\Lambda_{i,k,j} - \frac{\Delta\Lambda_{i,k,j}^2}{2\left(\Delta Y_{i,k,j} - \Delta Z_{i,j}\right)} - \frac{\gamma^2\left(\Delta Y_{i,k,j} - \Delta Z_{i,j}\right)}{2},\end{aligned}$$

$$\ln p\left(\Delta Z_{i,j}\right) = -\frac{1}{2}\ln(2\pi) + \ln \Delta\Lambda_{i,0,j} + \gamma \Delta\Lambda_{i,0,j} - \frac{3}{2}\ln \Delta Z_{i,j} - \frac{\Delta\Lambda_{i,0,j}^2}{2\Delta Z_{i,j}} - \frac{\gamma^2 \Delta Z_{i,j}}{2}.$$

EM algorithm

- **Initialization**: Start with initial values $\boldsymbol{\theta}^{(\mathbf{0})}$ for the parameters $\boldsymbol{\theta}$, and set the tolerance error ω.
- **E-step**: Calculate $Q\left(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(s)}\right) = \mathrm{E}\left[\ell(\boldsymbol{\theta} \mid \mathbb{Y}, \mathbb{Z}) \mid \mathbb{Y}, \boldsymbol{\theta}^{(s)}\right]$, based on the s-th iteration of parameters estimation $\boldsymbol{\theta}^{(s)}$.
- **M-step**: Compute the $(s+1)$-th parameter estimation $\boldsymbol{\theta}^{(s+1)}$ using $\boldsymbol{\theta}^{(s+1)} = \arg\max_{\boldsymbol{\theta}} Q\left(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(s)}\right)$.
- **Iteration**: Iterate through the E-step and M-step until $\left\|\boldsymbol{\theta}^{(s+1)} - \boldsymbol{\theta}^{(s)}\right\| < \omega$, where $\|\cdot\|$ denotes the Euclidean distance.
- **Output**: Obtain the ML estimates of $\boldsymbol{\theta}$.

Determine initial parameter estimators

1. Based on $\Delta\bar{Y}_{:,k,j} = 1/n\sum_{i=1}^{n}\Delta Y_{i,k,j}$, $\Delta s^2_{:,k,j} = \sum_{i=1}^{n}\left(\Delta Y_{i,k,j} - \Delta\bar{Y}_{:,k,j}\right)^2/(n-1)$, we calculate the estimate for γ:

$$\hat{\gamma} = \sqrt{\frac{\sum_{k=1}^{K}\sum_{j=1}^{m_i}\Delta\bar{Y}_{:,k,j}}{\sum_{k=1}^{K}\sum_{j=1}^{m_i}\Delta s^2_{:,k,j}}}.$$

2. Assuming $\hat{\gamma}$ is known, we optimize the formula to estimate $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$.

$$\begin{aligned}\Psi &= \arg\min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{k=1}^{K}\frac{\left(\mathrm{E}\left[\Delta Y_{i,k,j}\right]-\Delta Y_{i,k,j}\right)^2}{\mathrm{var}\left[\Delta Y_{i,k,j}\right]}\\ &= \arg\min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\hat{\gamma}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{k=1}^{K}\left[\frac{\hat{\gamma}^2\Delta Y^2_{i,k,j}}{\Delta\Lambda_{i,0,j}+\Delta\Lambda_{i,k,j}}+\Delta\Lambda_{i,0,j}+\Delta\Lambda_{i,k,j}\right].\end{aligned}$$

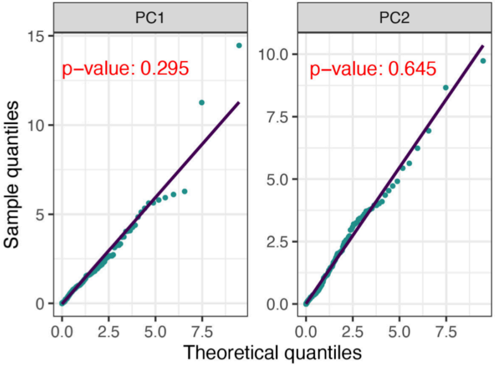
Model validation

- **Goodness of fit (GOF) test**: Evaluates each PC's rIG model using χ^2_1 Q-Q plots and the KS test based on the statistic $[\hat{\gamma}\Delta Y_{i,j,k} - \Delta\hat{\Lambda}_k(t_{i,j}) - \Delta\hat{\Lambda}_0(t_{i,j})]^2/\Delta Y_{i,j,k}$, which approximates an i.i.d. χ^2_1 distribution.
- **Common dispersion parameter γ test**: Analyzes if all PCs operate under a common γ or distinct γ_i for each PC through the chi-square test statistic $\tau = -2(\ell_1 - \ell_2)$, contrasting the log-likelihoods of a unified model against a heterogeneous model.
- **Model selection**: Uses the Akaike Information Criterion (AIC), $\mathrm{AIC} = 2\kappa - 2\ell$, to determine the most suitable model, focusing on the trade-off between model fit and complexity.

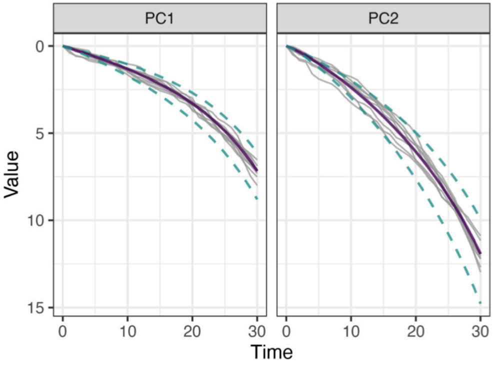
PMB degradation data

Table 3: Parameter point estimates regarding the PMB data.

Model	Scen.	α_0	α_1	α_2	β_1	β_2	γ	AIC
Proposed	I	0.866	1.296	1.463	0.028	0.124	3.030	-2219.427
	II	0.724	0.104	0.068	0.942	6.182	4.375	-2494.123
	III	0.100	0.994	1.205	0.395	0.531	4.263	**-2603.588**
	IV	0.098	0.009	0.025	42.099	30.476	4.299	-2594.714
Independent	Power	-	1.518	1.456	0.151	0.317	3.703	-637.266
	Log-linear	-	0.056	0.054	6.830	11.944	4.079	-744.887

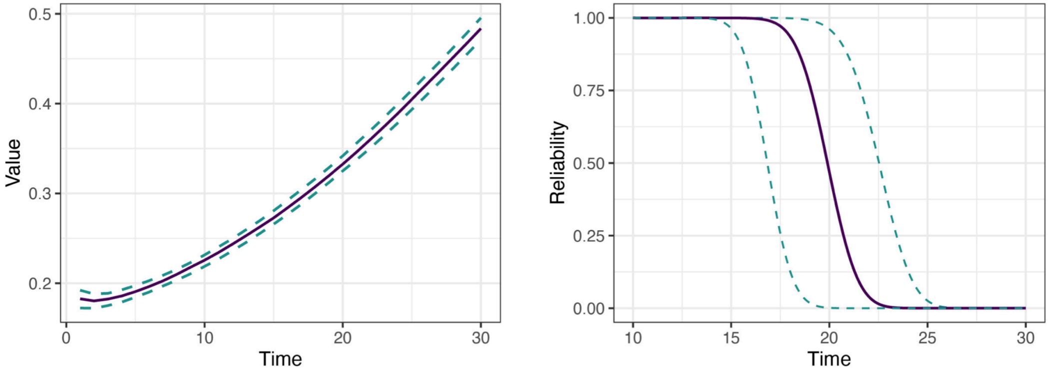


(a) Q-Q plots under scenario III model



(b) Estimated mean degradation path

Figure 8: Summary of PMB data analysis results: Q-Q plots under scenario III model and the estimated mean degradation path.



(a) Correlation coefficients (b) Reliability curves

Figure 9: Correlation coefficients and reliability curves for PMB data.

Fatigue crack-size data

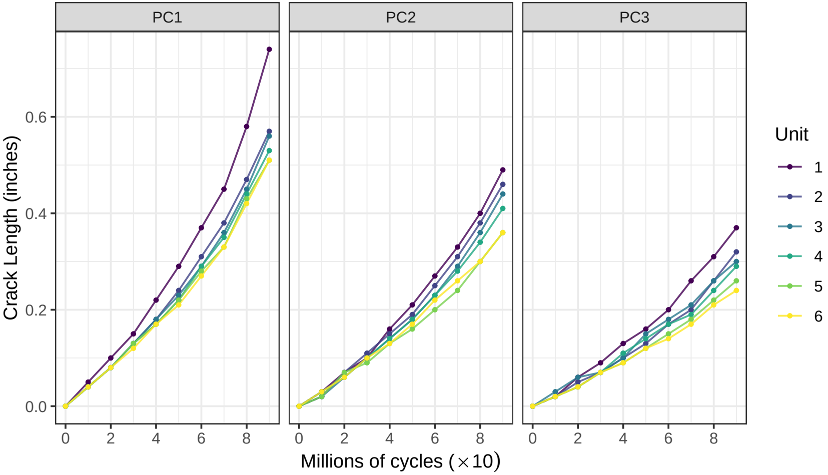


Figure 10: Degradation paths for fatigue crack-size growth data.

Table 4: Parameter point estimates regarding the fatigue crack-size data.

Model	Scen.	α_0	α_1	α_2	α_3	β_1	β_2	β_3	γ	AIC
Dependent	I	1.178	1.327	1.332	0.736	0.796	0.415	0.249	4.836	-717.838
	II	0.957	0.155	0.161	0.162	9.828	6.094	3.429	6.648	-804.636
	III	0.249	1.201	1.153	0.946	1.999	1.490	1.310	6.412	-822.131
	IV	0.067	0.119	0.111	0.090	19.683	16.286	15.236	6.789	**-1410.667**
Independent	Power	-	1.479	1.359	1.206	1.129	1.119	1.107	5.254	-221.197
	Log-linear	-	0.126	0.105	0.081	17.880	17.698	17.978	6.602	-281.749

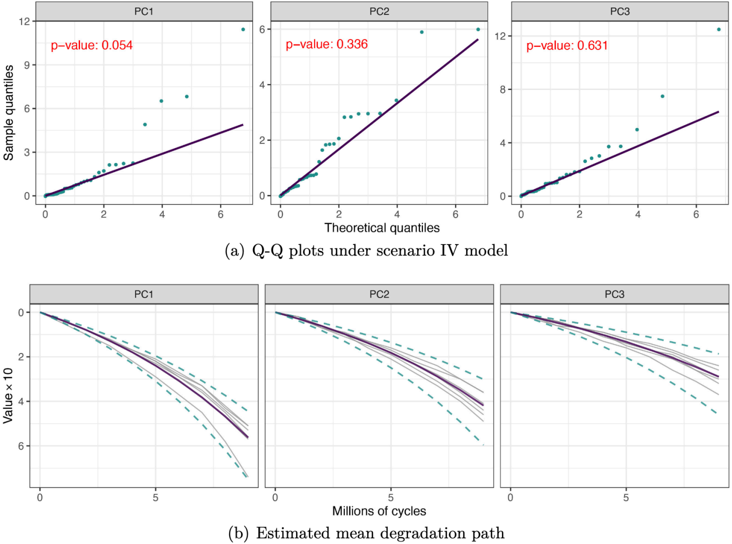


Figure 11: Summary of fatigue crack-size data analysis results: Q-Q plots under scenario IV and the estimated mean degradation path.

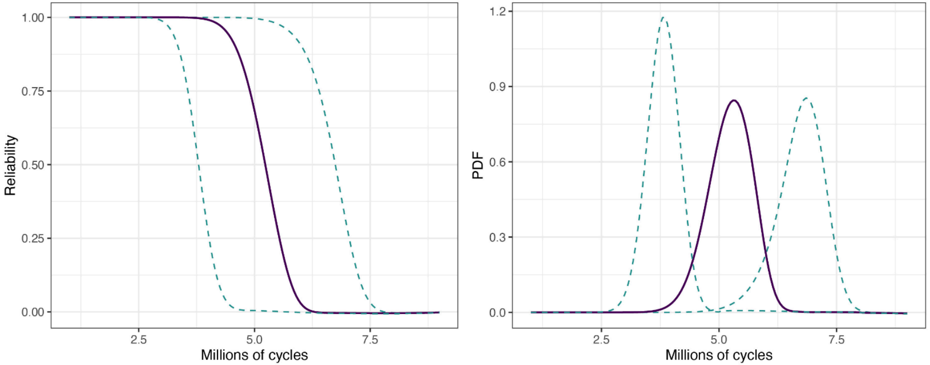


Figure 12: Reliability function and PDF for fatigue crack-size data.

Thanks!