Remaining Useful Life Prediction for Gamma Degradation Processes: A Recursive Bayesian Approach

徐安察

浙江工商大学统计与数学学院

Email: xuancha@mail.zjgsu.edu.cn

合作者：汪懿君，汤银才

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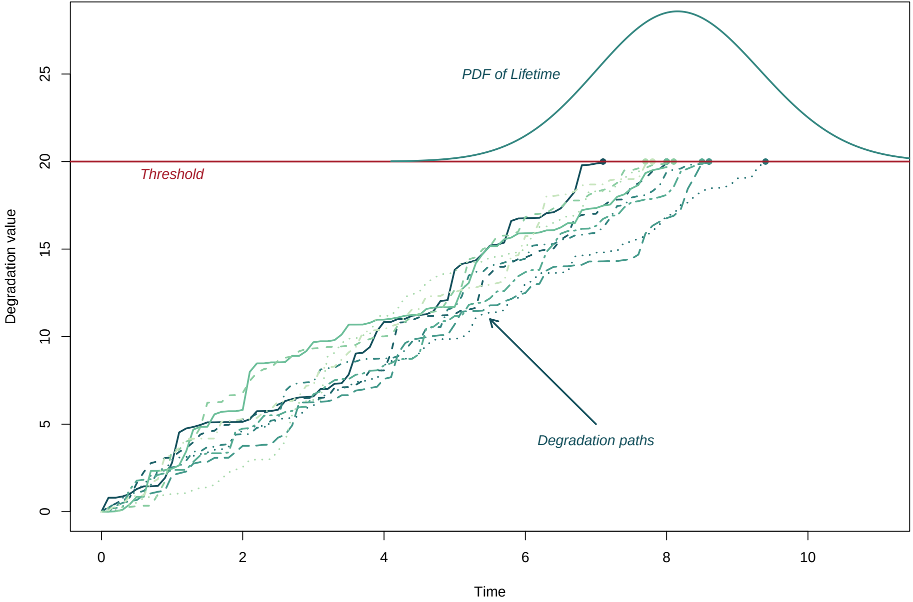
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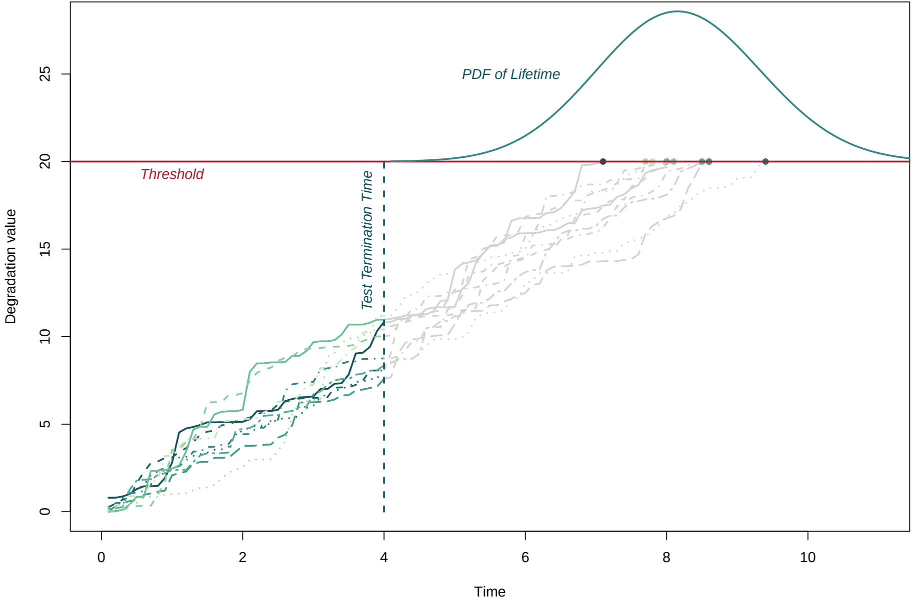
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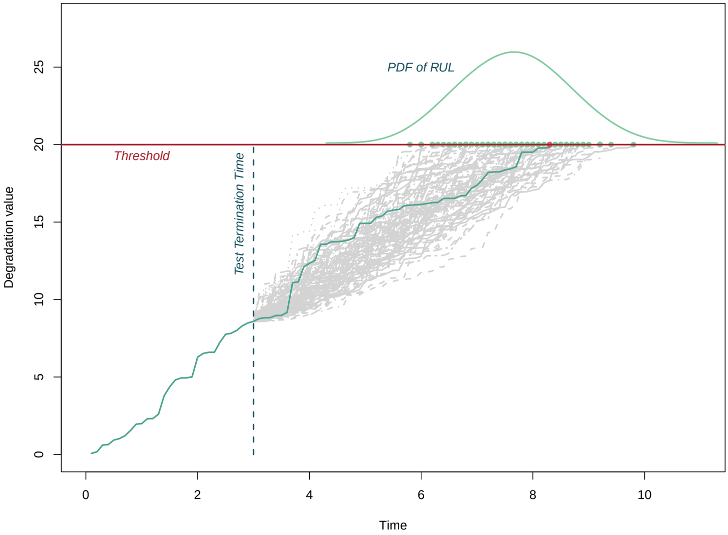
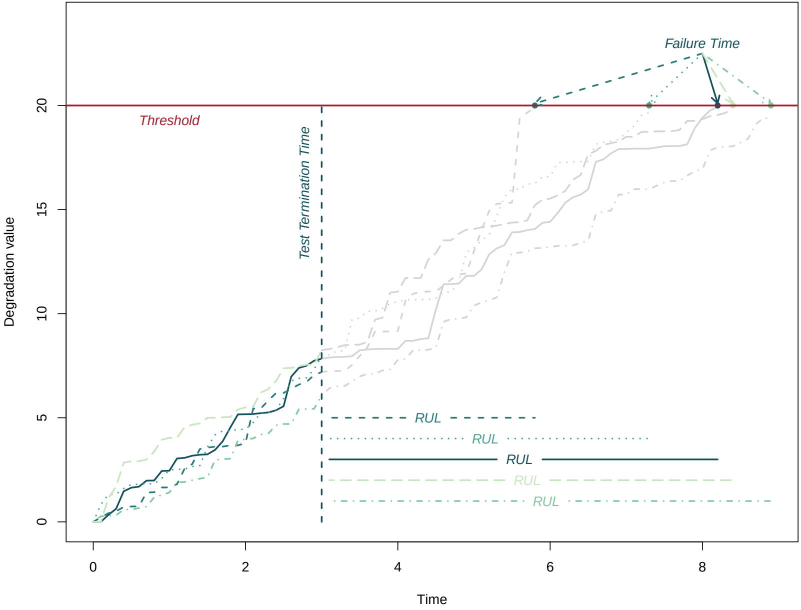
Degradation Models

- General path model.
- Stochastic process: Wiener, gamma, inverse Gaussian (IG), variance gamma, Ornstein–Uhlenbeck, etc.
- Review papers: Si et al. (2011), Ye and Xie (2015), Zhang et al. (2018).

Remaining useful life (RUL)

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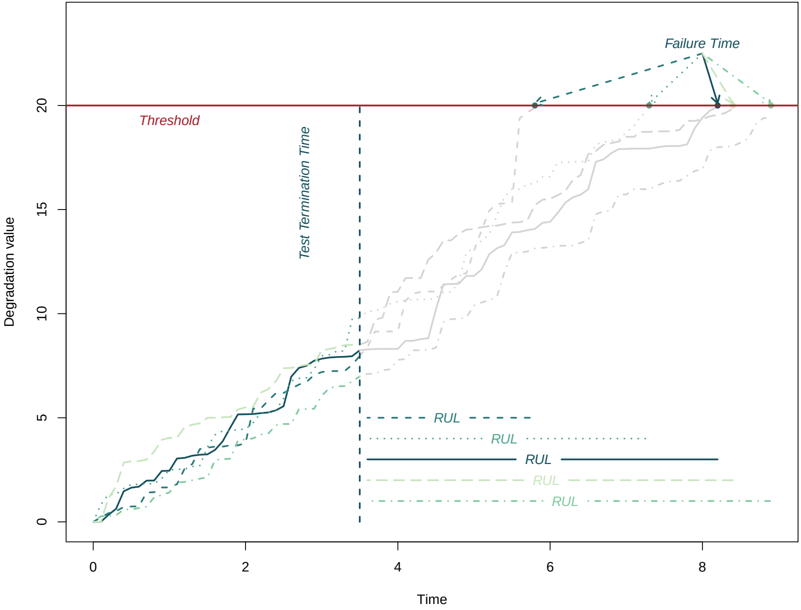

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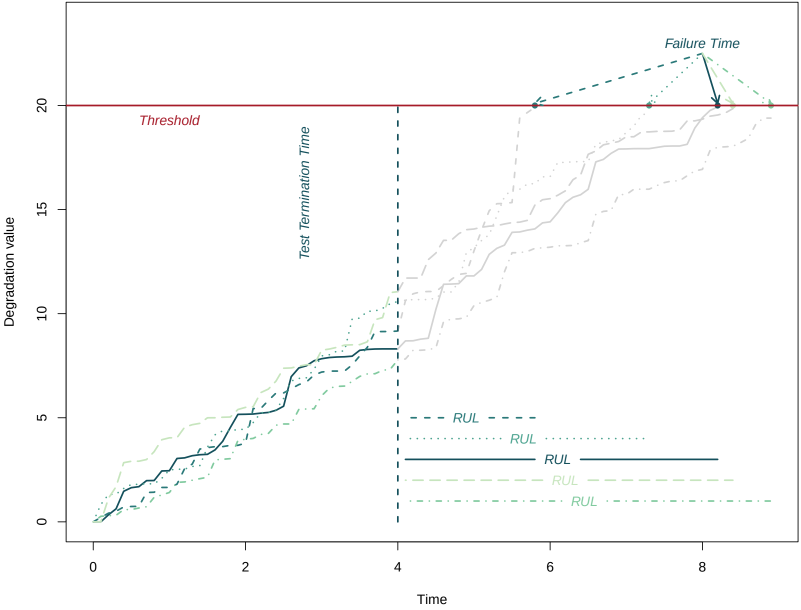
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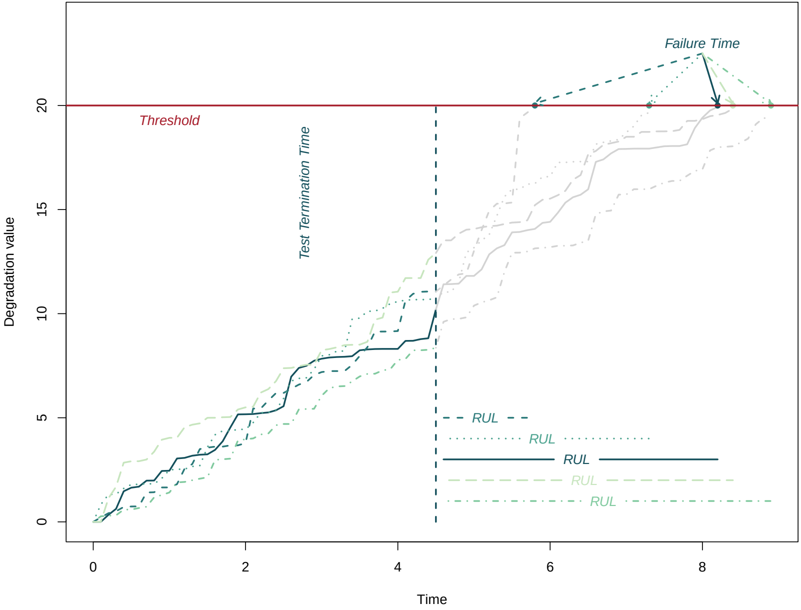
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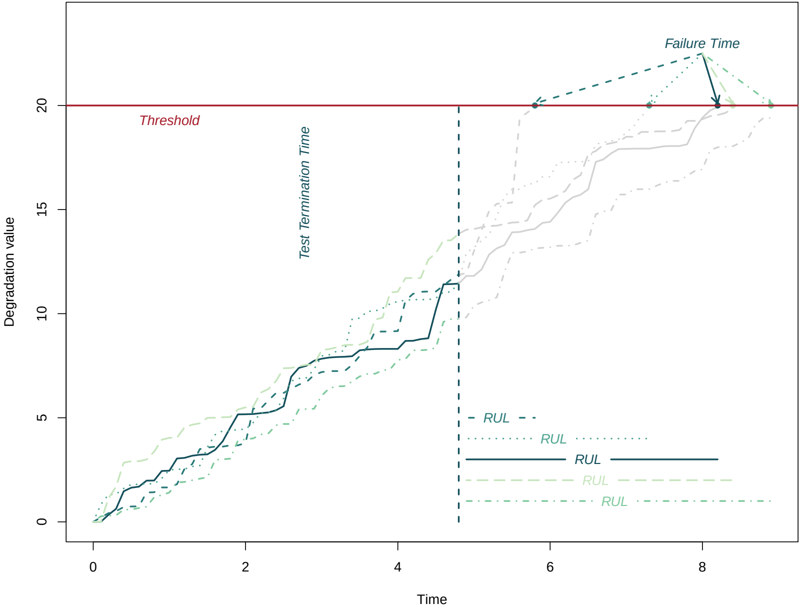
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Literature review

- For Wiener-based degradation models, the Kalman filter or its extended methods are often utilized to conduct online RUL predictions (Wang et al, 2011; Si et al., 2013; Wang and Tsui, 2018; Zhang et al., 2018).
- For gamma process, Paroissin (2017) and Xu and Shen (2018) developed recursive linear estimators for the mean and variance of the gamma process.
  - Using the same techniques for RUL prediction and interval estimation remains challenging.
  - As new observations emerge, conducting statistical analysis mandates a reiteration for the updated dataset, posing challenges with growing sample sizes in terms of data storage and computational efficiency.

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Gamma process

Definition

Gamma process $\{\mathcal{Y}(t), t \geq 0\}$ satisfies the following properties:

i) $\mathcal{Y}(0) = 0$ with probability 1;

ii) $\{\mathcal{Y}(t), t \geq 0\}$ has independent increments;

iii) The increment $\Delta Y_t = \mathcal{Y}(t) - \mathcal{Y}(s)$ follows gamma distribution ($Ga(\alpha(t-s), \beta)$) with probability density function (PDF)

$$f(y|\alpha, \beta) = \frac{\beta^{\alpha(t-s)} y^{\alpha(t-s)-1}}{\Gamma(\alpha(t-s))} \exp\{-\beta y\}, t > s,$$

where $\Gamma(\cdot)$ denotes the gamma function.

We denote the gamma process $\{\mathcal{Y}(t), t \geq 0\}$ as $\mathcal{GP}(\alpha t, \beta)$.

Lifetime

- Let $\mathcal{C}$ denote the threshold level for the degradation path.
- Lifetime of the system is defined as $\mathcal{T} = \inf\{t|\mathcal{Y}(t) \geq \mathcal{C}\}$.
- The reliability function of T is $R_{\mathcal{T}}(t|\alpha, \beta) = P(\mathcal{T} \geq t) = P(\mathcal{Y}(t) < \mathcal{C})$.

Data

- Degradation of system's performance characteristic follows gamma process $\mathcal{GP}(\alpha t, \beta)$.
- n systems from population are randomly selected and tested.
- Assume that the measurement time epochs are $t_1 < t_2 < \cdots < t_m$, and the corresponding degradation value of the i-th system at time epoch t_j is Y_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$.
- For the sake of simplifying notations, we assume that the time intervals between measurements are equal. That is, the measurement time epoch $t_j = jl$.
- Let $y_{ij} = Y_{ij} - Y_{ij-1}$, where $Y_{i0} = 0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.
- Then we have $y_{ij} \sim Ga(\alpha l, \beta)$.
- Denote the observed data as $\boldsymbol{y} = \{y_{ij}, i = 1, \ldots, n,\ j = 1, \ldots, m\}$.

Likelihood

Based on the data $\boldsymbol{y}$, the likelihood function is

$$
\begin{aligned}
L(\boldsymbol{y}|\alpha,\beta) &= \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{\beta^{\alpha l}}{\Gamma(\alpha l)} y_{ij}^{\alpha l-1}\exp\{-\beta y_{ij}\} \\
&\propto \frac{\beta^{mnl\alpha}}{[\Gamma(\alpha l)]^{mn}}\bar{y}_g^{mnl\alpha}\exp\{-mn\bar{y}_a\beta\},
\end{aligned} \tag{1}
$$

where $\bar{y}_g = \left[\prod_{i=1}^{n}\prod_{j=1}^{m} y_{ij}\right]^{1/(mn)}$ and $\bar{y}_a = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} y_{ij}$.

Conjugate prior

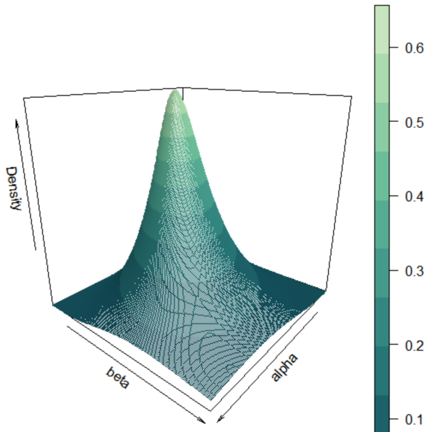
Theorem 1

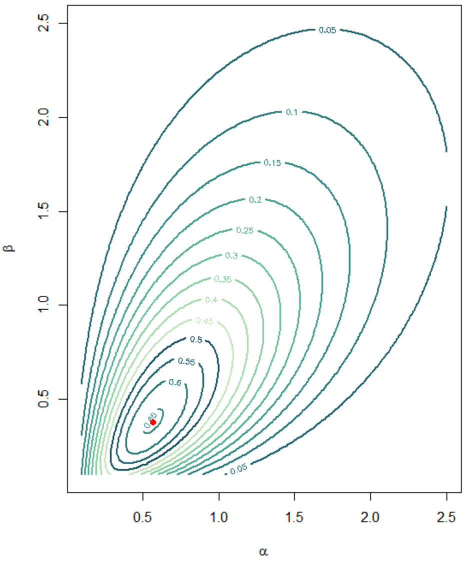
Given the likelihood function (1), *the conjugate prior for* α *and* β *is*

$$\pi(\alpha, \beta) = C\frac{(\beta\omega)^{\delta l\alpha}}{[\Gamma(l\alpha)]^{\delta}}\exp\{-\delta\lambda\beta\}, \tag{2}$$

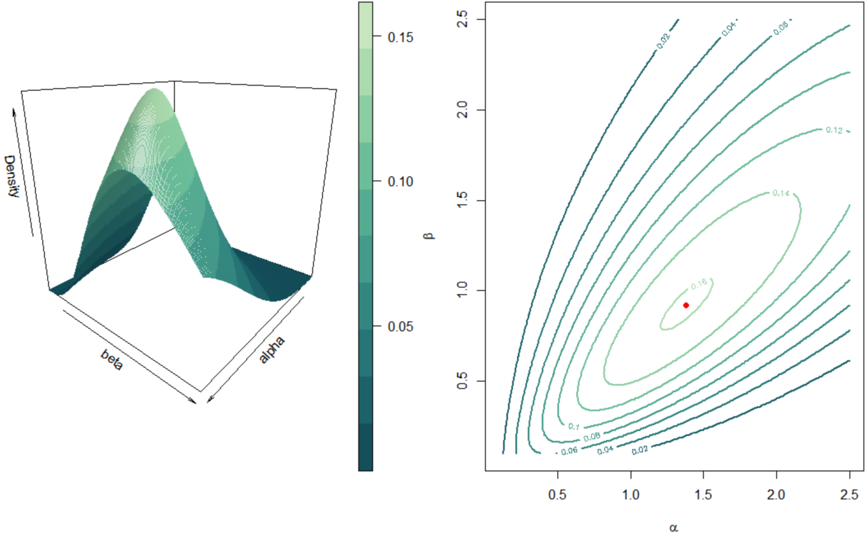
where C *is a normalized constant,* δ, ω *and* λ *are hyperparameters with nonnegative values, which describe kurtosis, shape and scale of the distribution, respectively.*

Visualization of conjugate prior $(\delta = 2, \omega = 0.5, \lambda = 1.5)$

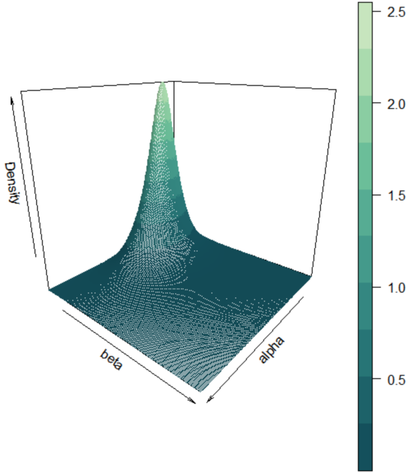


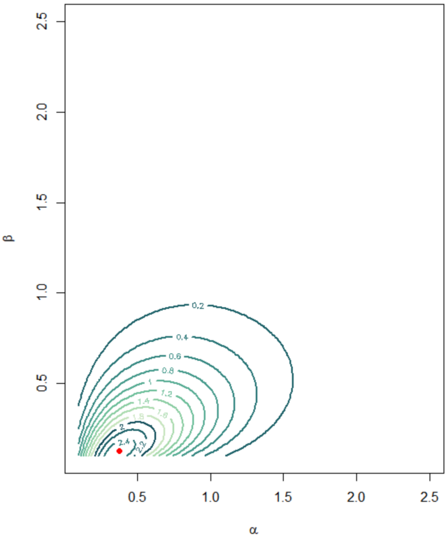


Visualization of conjugate prior ($\delta = 2, \omega = 1, \lambda = 1.5$)

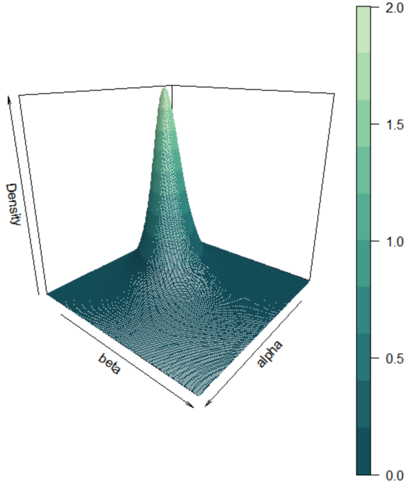


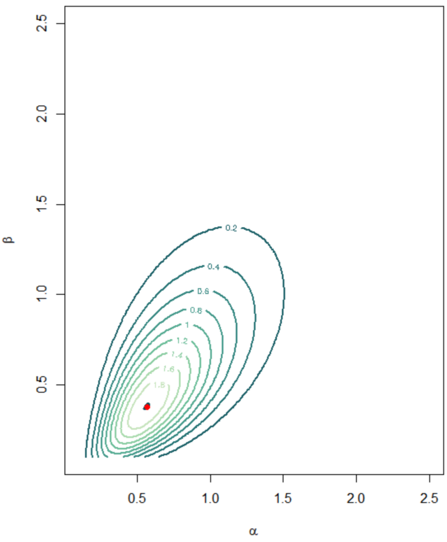
Visualization of conjugate prior ($\delta = 2, \omega = 0.5, \lambda = 3$)





Visualization of conjugate prior $(\delta = 5, \omega = 0.5, \lambda = 1.5)$





Determination of hyperparameters

Choosing the values of hyperparemeters according to amount of prior information

- Informative priors: large δ, small ω, or large λ.
- Diffuse priors: small δ, large ω, or small λ.

The joint prior distribution (2) can be written as

$$\pi(\alpha,\beta) = \pi(\beta|\alpha)\pi(\alpha)$$
$$\propto \frac{(\delta\lambda)^{\delta l\alpha+1}\beta^{\delta l\alpha}}{\Gamma(1+\delta l\alpha)}\exp\{-\delta\lambda\beta\}\cdot\frac{\Gamma(1+\delta l\alpha)}{[\Gamma(l\alpha)]^{\delta}}\exp\left\{-\alpha\delta l\log\left(\frac{\delta\lambda}{\omega}\right)\right\}.$$

- $\beta|\alpha \sim Ga(1+\delta l\alpha, \delta\lambda)$.
- The marginal density of α is proportional to

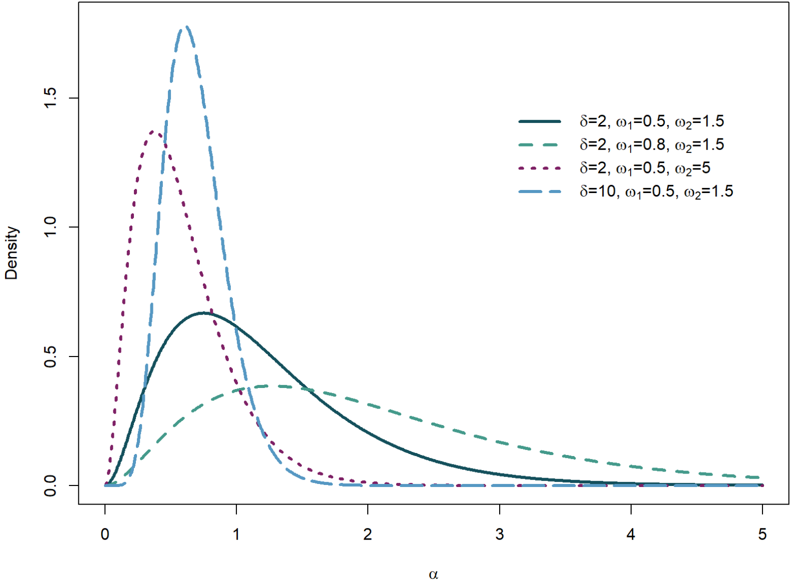
$$h(\alpha) = \frac{\Gamma(1+\delta l\alpha)}{[\Gamma(l\alpha)]^{\delta}}\exp\left\{-\alpha\delta l\log\left(\frac{\delta\lambda}{\omega}\right)\right\}. \tag{3}$$

- Using Stirling's formula and $\alpha \to \infty$,

$$\frac{\Gamma(1+\delta l \alpha)}{[\Gamma(l\alpha)]^{\delta}} \equiv O\left(\alpha^{(\delta+1)/2} \exp\{\alpha \delta l \log(\delta)\}\right).$$

- When $\alpha \to \infty$, $h(\alpha) \equiv O\left(\alpha^{(\delta+1)/2} \exp\left\{-\alpha\delta l \log\left(\frac{\lambda}{\omega}\right)\right\}\right)$.
- Thus, to make the conjugate prior $\pi(\alpha, \beta)$ proper, the condition $\omega < \lambda$ should be satisfied.
- Then the tail of $\pi(\alpha)$ can be approximated by $Ga\left(\frac{\delta+3}{2}, \delta l \log\left(\frac{\lambda}{\omega}\right)\right)$.
- We call $\pi(\alpha, \beta)$ approximated-gamma-gamma distribution, denoted as $AGG(\delta, \omega, \lambda)$.

Marginal prior of α



Generate random numbers from $AGG(\delta, \omega, \lambda)$

Algorithm 1: Gibbs sampling (GS)

1. $\beta|\alpha \sim Ga(1+\delta l\alpha, \delta\lambda)$.
2. Given β, the conditional density of α is proportional to $\frac{(\beta\omega)^{\delta l\alpha}}{[\Gamma(l\alpha)]^{\delta}}$, which is log-concave.

Generation of random numbers from $AGG(\delta, \omega, \lambda)$

Algorithm 2: Discrete grid sampling (DGS)

Discrete grid sampling is utilized for generating random numbers from $\pi(\alpha)$ approximately.

1. Choose an interval (A_1, A_2), in which the probability that α lies in is nearly 1.
2. Select M grids equally lies in the interval (A_1, A_2), and compute the unnormalized marginal distribution of α (3) on the grids.
3. Having computed the relative posterior density at a grid, we normalize by approximating $\pi(\alpha)$ as discrete distribution over the grids and setting the total probability in the grids to 1.
4. Generate random numbers of α from the normalized discrete distribution.
5. Given α, generate β from $Ga(1 + \delta l\alpha, \delta\lambda)$.

Generate random numbers from $AGG(\delta, \omega, \lambda)$

Algorithm 3: Sampling importance resampling (SIR)

1. Choose $Ga(a, b)$ as the instrumental distribution.
2. The values of a and b can be determined as follows.
   - Let $\tilde{\alpha} = \arg\max_{\alpha} \log h(\alpha)$ and $I(\tilde{\alpha}) = \left.\frac{\partial^2 \log h(\alpha)}{\partial \alpha^2}\right|_{\alpha=\tilde{\alpha}}$.
   - Initialize b as $b_0 = \delta l \log\left(\frac{\lambda}{\omega}\right)$ and a as $a_0 = \tilde{\alpha} b_0$.
   - Compute the precision ratio $R = \frac{b_0^2/a_0}{I(\tilde{\alpha})}$, and update $a = a_0/R$ and $b = b_0/R$.
3. Then generate M random numbers from $Ga(a, b)$, and the weight of each number can be computed by function $h(\alpha)/f_{Ga}(\alpha|a, b)$, where $f_{Ga}(\alpha|a, b)$ denotes PDF of $Ga(a, b)$.
4. Resampling α with replacement from the weighted M random numbers.
5. Given α, generate β from $Ga(1 + \delta l \alpha, \delta\lambda)$.

Posterior distribution

Theorem 2

Given the likelihood function (1) *and prior* $\pi(\alpha,\beta)$ (2), *the joint posterior distribution of* α *and* β *is*

$$AGG\left(mn+\delta,\ \bar{y}_g^{\frac{mn}{mn+\delta}}\omega^{\frac{\delta}{mn+\delta}},\ \frac{mn\bar{y}_m+\delta\lambda}{mn+\delta}\right).$$

- Special values of hyperparameters ω and λ: $\omega=\bar{y}_g,\ \lambda=\bar{y}_m$. Then the joint posterior is $AGG\left(mn+\delta,\ \bar{y}_g,\ \bar{y}_m\right)$.
- In this setting, the hyperparameter δ behaves like number of measurements. The value of δ can be determined according to measurement-equivalent of the amount of information.

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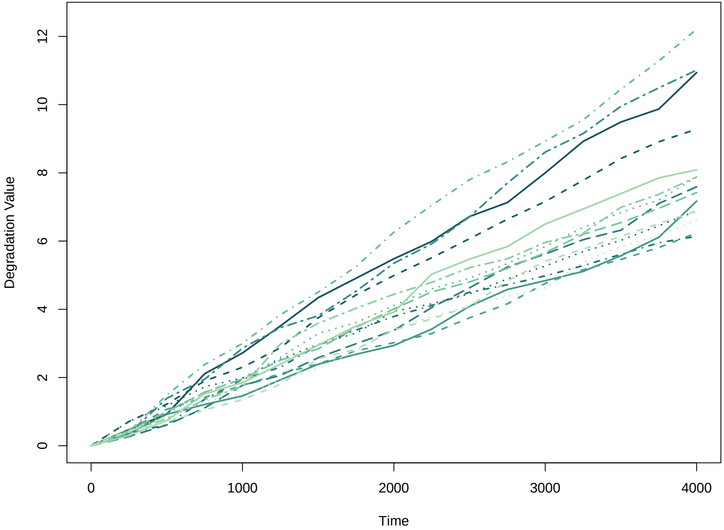
Laser degradation data (Meeker & Escobar , 1998)



Estimates

- Prior: $AGG\left(1,\ \bar{y}_g,\ \bar{y}_m\right)$.
- Parameters: α, β and reliability at time 4500 hours $R(4500)$.
- Gibbs sampling: 3000 iterations with the first 1000 burn-in sample
- Discrete grid sampling: 10000 grids in the interval (0,10), and finally generate posterior sample with sample size 1000.
- SIR: 10000 random numbers from instrumental distribution, and resampling posterior sample with sample size 1000.

Table 1: The point and 95% CI estimates of α, β and $R(4500)$.

Estimate	GS			DGS			SIR		
	α	β	$R(4500)$	α	β	$R(4500)$	α	β	$R(4500)$
Point	0.0309	15.342	0.879	0.0308	15.325	0.878	0.0310	15.438	0.882
2.5%	0.0258	12.693	0.740	0.0260	12.698	0.737	0.0256	12.677	0.743
97.5%	0.0366	18.332	0.963	0.0370	18.328	0.962	0.0366	18.368	0.964

Simulation settings

- $\alpha = 0.031$, $\beta = 15.35$, $R(4500) = 0.88$, and mean-time-to-failure (MTTF) =4976.74.
- $m = 16$, $n = 15$, $l = 250$.
- Hyperparameters $\delta = 0,\ 1$, $m/4$ and $m/2$; $\omega = \bar{y}_g$; $\lambda = \bar{y}_m$.
- 10000 repetitions for comparing three algorithms.
- Indexes of assessing different algorithms: absolute relative error (ARB) and squared root of mean squared error (RMSE) of Bayesian point estimates, frequentist coverage probability (FCP) of 95% credible interval, computational time.

Table 2: ARBs of point estimates of the parameters.

Algorithm	$\delta = 0$				$\delta = 1$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.0245	0.0256	0.0161	0.00109	0.0243	0.0254	0.0161	0.00108
DGS	0.0245	0.0256	0.0161	0.0011	0.0245	0.0256	0.0161	0.00109
SIR	0.0245	0.0256	0.0161	0.00109	0.0245	0.0256	0.0161	0.00109
Algorithm	$\delta = \frac{m}{4}$				$\delta = \frac{m}{2}$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.0233	0.0247	0.0153	0.00136	0.0234	0.0248	0.0151	0.00137
DGS	0.0234	0.0249	0.0152	0.00137	0.0233	0.0247	0.0151	0.00136
SIR	0.0234	0.0249	0.0152	0.00138	0.0232	0.0246	0.0151	0.00136

Table 3: RMSEs of point estimates of the parameters.

Algorithm	$\delta = 0$				$\delta = 1$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.00292	1.491	0.0573	113.28	0.00291	1.489	0.0574	113.25
DGS	0.00290	1.484	0.0574	113.22	0.00289	1.482	0.0573	113.19
SIR	0.00289	1.483	0.0573	113.21	0.00289	1.482	0.0574	113.20
Algorithm	$\delta = \frac{m}{4}$				$\delta = \frac{m}{2}$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.00289	1.484	0.0569	113.68	0.00290	1.486	0.0568	113.70
DGS	0.00286	1.479	0.0569	113.63	0.00287	1.475	0.0567	113.59
SIR	0.00284	1.479	0.0568	113.60	0.00286	1.476	0.0567	113.55

Table 4: Lengths of 95% credible intervals of the parameters.

Algorithm	$\delta = 0$				$\delta = 1$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.0109	5.588	0.224	447.444	0.0109	5.585	0.224	446.609
DGS	0.0109	5.629	0.224	446.274	0.0109	5.620	0.223	445.596
SIR	0.0110	5.630	0.224	446.414	0.0109	5.624	0.223	445.318
Algorithm	$\delta = \frac{m}{4}$				$\delta = \frac{m}{2}$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.0108	5.541	0.222	444.082	0.0107	5.502	0.220	440.493
DGS	0.0108	5.582	0.221	443.013	0.0107	5.535	0.219	439.562
SIR	0.0109	5.581	0.221	443.131	0.0108	5.536	0.219	439.373

Table 5: Frequentist coverage probabilities of 95% credible intervals of the parameters.

Algorithm	$\delta = 0$				$\delta = 1$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.938	0.937	0.9434	0.9447	0.938	0.9359	0.9444	0.9464
DGS	0.9364	0.9417	0.9441	0.9463	0.936	0.9401	0.9438	0.9451
SIR	0.9454	0.9412	0.9441	0.9442	0.9446	0.9429	0.9436	0.9438
Algorithm	$\delta = \frac{m}{4}$				$\delta = \frac{m}{2}$			
	α	β	$R(4500)$	MTTF	α	β	$R(4500)$	MTTF
GS	0.9384	0.9366	0.9456	0.9476	0.9328	0.9342	0.9431	0.9444
DGS	0.9318	0.944	0.9449	0.9463	0.9276	0.9398	0.9419	0.9439
SIR	0.9433	0.9433	0.9433	0.9466	0.9404	0.9413	0.9428	0.9428

- The computational time of the three algorithms for each sample are 0.602, 0.00341 and 0.00499 seconds in a desktop with Intel(R) Core(TM) i7-10700 CPU at 2.9 GHz and 16 GB RAM running under a Windows 11 operating system.

目录

Data

- Assume that heterogeneity exists among systems. The degradation of the i-th system's performance characteristic follows gamma process $\mathcal{GP}(\alpha t, \beta_i)$.
- Let Y_{ij} be the degradation value of the i-th system at time epoch $t_j = jl$, $i = 1, \ldots, n,\ j = 1, \ldots, m$.
- Let $y_{ij} = Y_{ij} - Y_{ij-1}$, where $Y_{i0} = 0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.
- Then we have $y_{ij} \sim Ga(\alpha l, \beta_i)$.
- Denote the observed data as $\boldsymbol{y}_{(\boldsymbol{m})} = \{y_{ij}, i = 1, \ldots, n,\ j = 1, \ldots, m\}$.

Likelihood

Based on the data $\boldsymbol{y_{(m)}}$, the likelihood function is

$$\begin{aligned} L(\boldsymbol{y_{(m)}}|\alpha, \beta_1, \ldots, \beta_n) &= \prod_{i=1}^{n}\prod_{j=1}^{m} \frac{\beta_i^{\alpha l}}{\Gamma(\alpha l)} y_{ij}^{\alpha l-1} \exp\{-\beta_i y_{ij}\} \\ &\propto \frac{\bar{\beta}_g^{mnl\alpha}}{[\Gamma(\alpha l)]^{mn}} \bar{y}_g^{mnl\alpha} \exp\left\{-\sum_{i=1}^{n} m\bar{y}_i\beta\right\}, \end{aligned} \quad (4)$$

where $\bar{\beta}_g = [\prod_{i=1}^{n}\beta_i]^{1/n}$ and $\bar{y}_i = \frac{1}{m}\sum_{j=1}^{m} y_{ij}$, $i = 1, \ldots, n$.

Conjugate prior

Theorem 3

Given the likelihood function (4), the conjugate prior for $(\alpha, \beta_1, \ldots, \beta_n)'$ *is*

$$\pi(\alpha, \beta_1, \ldots, \beta_n) = C \frac{\left(\bar{\beta}_g \omega\right)^{\delta_1 l \alpha}}{\left[\Gamma(l\alpha)\right]^{\delta_1}} \exp\left\{-\sum_{i=1}^{n} \delta_2 \lambda_i \beta_i\right\}, \tag{5}$$

where C *is a normalized constant,* δ_1, δ_2, ω *and* λ_i*s are hyperparameters with nonnegative values.*

Decomposition

$$
\begin{aligned}
\pi(\alpha, \beta_1, \ldots, \beta_n) &= \prod_{i=1}^{n} \pi(\beta_i|\alpha)\pi(\alpha) \\
&\propto \prod_{i=1}^{n} \frac{(\delta_2\lambda_i)^{1+\delta_1 l\alpha/n}\beta_i^{\delta_1 l\alpha/n}}{\Gamma(1+\delta_1 l\alpha/n)} \exp\{-\delta_2\lambda_i\beta_i\} \\
&\times \frac{[\Gamma(1+\delta_1 l\alpha/n)]^n}{[\Gamma(l\alpha)]^{\delta_1}} \exp\left\{-\alpha\delta_1 l\left[\log\left(\frac{\delta_2}{\omega}\right) + \frac{1}{n}\sum_{i=1}^{n}\log\lambda_i\right]\right\}.
\end{aligned}
$$

- $\beta_i|\alpha \sim Ga(1+\delta_1 l\alpha/n, \delta_2\lambda_i)$.
- The marginal density of α is proportional to

$$
g(\alpha) = \frac{[\Gamma(1+\delta_1 l\alpha/n)]^n}{[\Gamma(l\alpha)]^{\delta_1}} \exp\left\{-\alpha\delta_1 l\left[\log\left(\frac{\delta_2}{\omega}\right) + \frac{1}{n}\sum_{i=1}^{n}\log\lambda_i\right]\right\}. \tag{6}
$$

- Using Stirling's formula and $\alpha \to \infty$,

$$g(\alpha) \equiv O\left(\alpha^{\frac{\delta_1+n}{2}} \exp\{-K\alpha\}\right),$$

where $K = \delta_1 l \left[\log\left(\frac{n\delta_2}{\delta_1}\right) + \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{\lambda_i}{\omega}\right)\right]$.

- Thus, to make the conjugate prior $\pi(\alpha, \beta_1, \ldots, \beta_n)$ proper, the condition $K > 0$ should be satisfied.

- Then the tail of $\pi(\alpha)$ can be approximated by $Ga\left(\frac{\delta_1+n+2}{2}, K\right)$.

- We call $\pi(\alpha, \beta)$ approximated-gamma-multivariate-gamma distribution, denoted as $AGMG_n(\boldsymbol{\gamma}, \omega, \boldsymbol{\xi})$, where $\boldsymbol{\gamma} = (\delta_1, \delta_2)'$, and $\boldsymbol{\xi} = (\lambda_1, \ldots, \lambda_n)'$.

Sampling from $AGMG_n(\boldsymbol{\gamma},\ \omega,\ \boldsymbol{\lambda})$

Algorithm 4: SIR

1. Choose $Ga(a,b)$ as the instrumental distribution.
2. The values of a and b can be determined as follows.
   - Let $\tilde{\alpha} = \arg\max_{\alpha} \log g(\alpha)$ and $I(\tilde{\alpha}) = \left.\frac{\partial^2 \log g(\alpha)}{\partial \alpha^2}\right|_{\alpha=\tilde{\alpha}}$.
   - Initialize b as $b_0 = K$ and a as $a_0 = \tilde{\alpha} b_0$.
   - Compute the precision ratio $R = \frac{b_0^2/a_0}{I(\tilde{\alpha})}$, and update $a = a_0/R$ and $b = b_0/R$.
3. Then generate M random numbers from $Ga(a,b)$, and the weight of each number can be computed by function $g(\alpha)/f_{Ga}(\alpha|a,b)$.
4. Resampling α with replacement from the weighted M random numbers.
5. Given α, generate β_i from $Ga(1+\delta_1 l\alpha/n, \delta_2\lambda_i)$.

Posterior distribution

Theorem 4

Given the likelihood function (4) *and prior* (5)*, the joint posterior distribution of* $(\alpha, \beta_1, \ldots, \beta_n)'$ *is*

$$AGMG_n\left(\boldsymbol{\gamma}_{(m)},\ \omega_{(m)},\ \boldsymbol{\lambda}_{(m)}\right),$$

where $\boldsymbol{\gamma}_{(m)} = (mn + \delta_1, m + \delta_2)'$, $\omega_{(m)} = \bar{y}_{g(m)}^{\frac{mn}{mn+\delta_1}} \omega^{\frac{\delta_1}{mn+\delta_1}}$,
$\bar{y}_{g(m)} = \left[\prod_{i=1}^{n} \prod_{j=1}^{m} y_{ij}\right]^{\frac{1}{mn}}$, $\boldsymbol{\lambda}_{(m)} = \left(\frac{m\bar{y}_1+\delta_2\lambda_1}{m+\delta_2}, \ldots, \frac{m\bar{y}_n+\delta_2\lambda_n}{m+\delta_2}\right)'$.

- Special values of hyperparameters ω and λ_i: $\omega = \bar{y}_{g(m)},\ \lambda_i = \bar{y}_{i(m)}$ Then the joint posterior is $AGMG_n\left(\boldsymbol{\gamma}_{(m)},\ \bar{y}_{g(m)},\ (\bar{y}_{1(m)}, \ldots, \bar{y}_{n(m)})'\right)$.
- The hyperparameters δ_1 and δ_2 behave like number of measurements.

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RUL

- Assume that all the degradation values of the i-th system until time t_m are less than $\mathcal{C}$. The remaining useful life (RUL) of the i-th system at time t_m is defined as

$$Z_{it_m} = \inf\{z : \mathcal{Y}_i(z + t_m) \geq \mathcal{C} | \mathcal{Y}_i(t_m) < \mathcal{C}\}.$$

- The reliability function of Z_{it_m} is

$$R_{Z_{it_m}}(z|\alpha, \beta_i) = P(Z_{it_m} \geq z) = P(\mathcal{Y}_i(z + t_m) < \mathcal{C}).$$

Approximation

- The PDF of Z_{it_m}: $f_{Z_{it_m}}(z|\alpha,\beta_i) = -\frac{\partial R_{Z_{it_m}}(z|\alpha,\beta_i)}{\partial z}$, which is too complicated.
- Park and Padgett (2005) recommended a two-parameter Birnbaum–Saunders distribution $BS(\alpha^*,\beta_i^*)$ with CDF $\Phi\left(\frac{1}{\alpha_i^*}\left[\sqrt{\frac{z}{\beta_i^*}}-\sqrt{\frac{\beta_i^*}{z}}\right]\right)$ to approximate the distribution of Z_{it_m}, where $\alpha_i^* = \sqrt{\frac{1}{\beta_i(\mathcal{C}-Y_{im})}}$ and $\beta_i^* = \frac{\beta_i(\mathcal{C}-Y_{im})}{\alpha}$, $\Phi(\cdot)$ is the CDF of standard normal distribution.
- Then mean of Z_{it_m} can be approximated by

$$\mu_{im}(\alpha,\beta_i) = \beta_i^*\left(1+(\alpha_i^*)^2/2\right) = \frac{1+2\beta_i(\mathcal{C}-Y_{im})}{2\alpha}.$$

- The lower ρ-th quantile of the distribution of Z_{it_m} can be approximated by

$$\mu_{im}^{\rho}(\alpha,\beta_i) = \frac{\beta_i^*}{4}\left[u_\rho\alpha^* + \sqrt{(u_\rho\alpha^*)^2+4}\right]^2,$$

where u_ρ is the ρ-th quantile of the standard normal distribution.

RUL prediction

- Bayesian point prediction of RUL of the i-th system at time t_m:

$$\tilde{\mu}_{im} = \int_0^{\infty}\int_0^{\infty} \mu_{im}(\alpha, \beta_i)\pi(\alpha, \beta_i|\boldsymbol{y_{(m)}})\mathrm{d}\alpha\mathrm{d}\beta_i. \quad (7)$$

- Bayesian interval prediction of RUL of the i-th system at time t_m with $1-\rho$ credible level:

$$\left(\tilde{\mu}_{im}^{\rho/2}, \tilde{\mu}_{im}^{1-\rho/2}\right), \quad (8)$$

where $\tilde{\mu}_{im}^{\rho} = \int_0^{\infty}\int_0^{\infty} \mu_{im}^{\rho}(\alpha, \beta_i)\pi(\alpha, \beta_i|\boldsymbol{y_{(m)}})\mathrm{d}\alpha\mathrm{d}\beta_i$.

Procedure of online RUL prediction

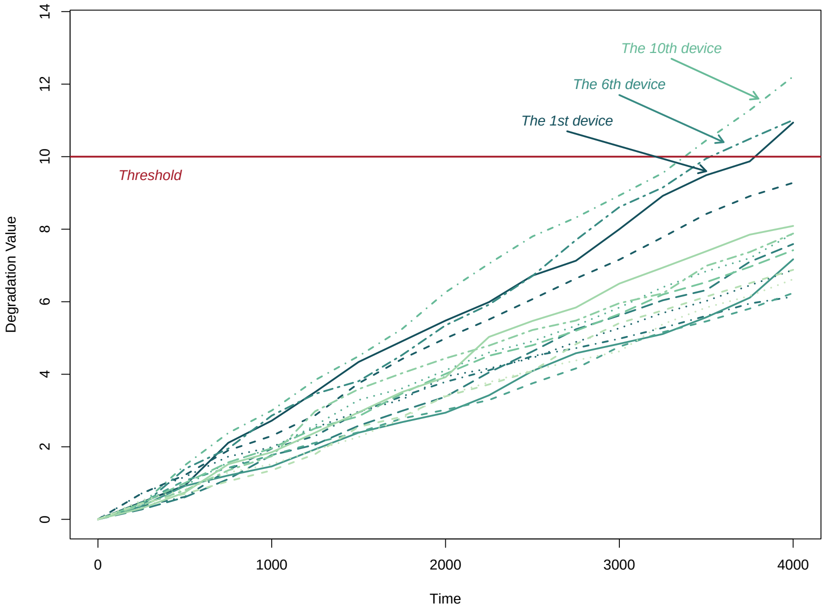
1. Collect new observations $(y_{1m+1}, \ldots, y_{nm+1})$ at time $t_{m+1} = (m+1)l$
2. Update the hyperparameters in the posterior distribution of $(\alpha, \beta_1, \ldots, \beta_n)'$ iteratively:

$$\gamma_{(m+1)} = \gamma_{(m)} + (n, 1)', \omega_{(m+1)} = \omega_{(m)}^{\frac{mn+\delta_1}{(m+1)n+\delta_1}} \left[\prod_{i=1}^{m} y_{im+1} \right]^{\frac{1}{(m+1)n+\delta_1}},$$

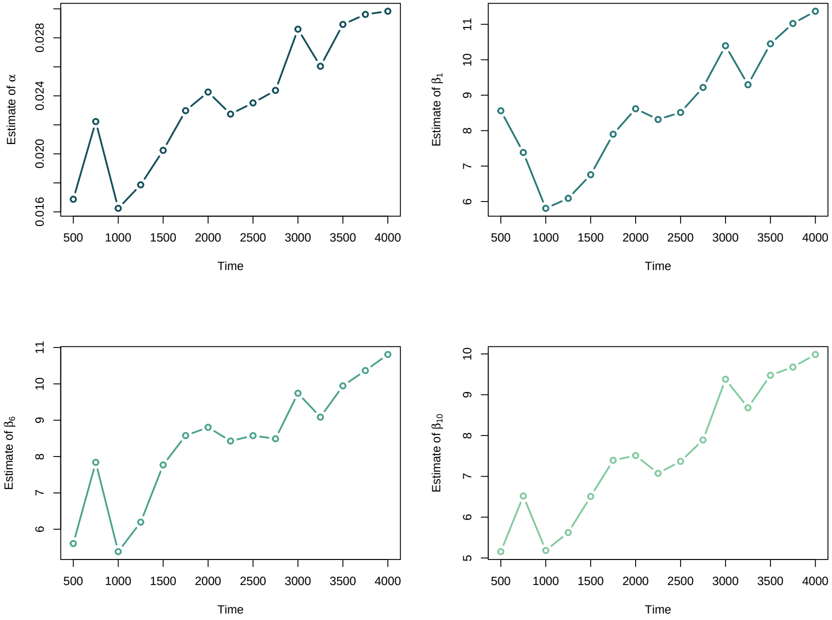
$$\boldsymbol{\lambda}_{(m+1)} = \frac{m+\delta_2}{m+1+\delta_2} \boldsymbol{\lambda}_{(m)} + \frac{1}{m+1+\delta_2} (y_{1m+1}, \ldots, y_{nm+1})'.$$

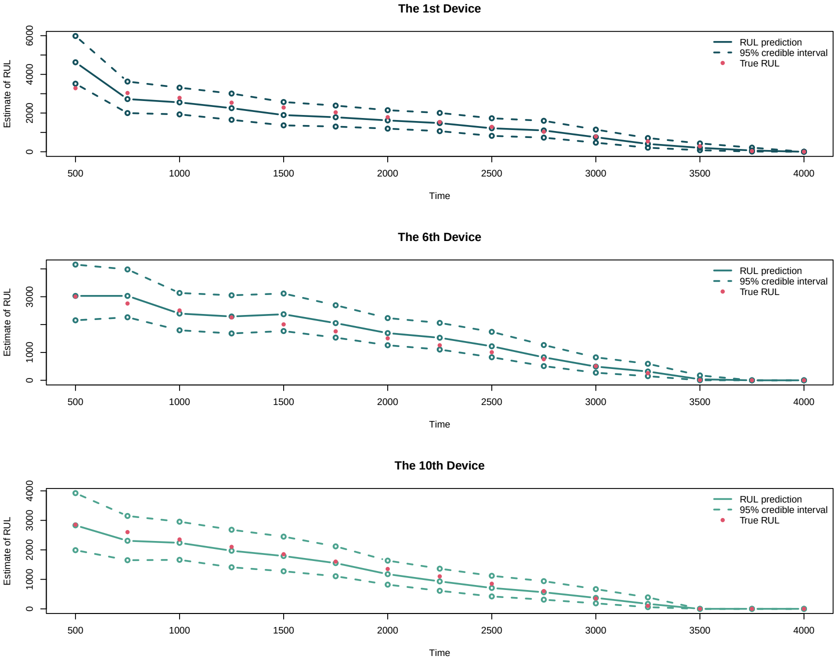
3. Generate posterior sample of $(\alpha, \beta_1, \ldots, \beta_n)'$ by algorithm 4.
4. Evaluate (7) and (8) by posterior sample using Monte Carlo integration, and thus obtain the Bayesian point and interval prediction of RUL at time t_{m+1}.

目录


The 10th device
The 6th device
The 1st device
Threshold
Degradation Value
Time
0
2
4
6
8
10
12
14
0
1000
2000
3000
4000

- Using linear interpolation method, we can obtain the true failure time for the first, sixth and tenth devices, which are 3785.75, 3506.75 and 3351.25 hours, respectively.
- Prediction of RUL of the three devices starts from the second measurement (500 hours).


Estimate of α
0.016
0.020
0.024
0.028
500
1000
1500
2000
2500
3000
3500
4000
Time
Estimate of β_1
6
7
8
9
10
11
500
1000
1500
2000
2500
3000
3500
4000
Time
Estimate of β_6
6
7
8
9
10
11
500
1000
1500
2000
2500
3000
3500
4000
Time
Estimate of β_{10}
5
6
7
8
9
10
500
1000
1500
2000
2500
3000
3500
4000
Time


The 1st Device
Estimate of RUL
0
2000
4000
6000
500
1000
1500
2000
2500
3000
3500
4000
Time
RUL prediction
95% credible interval
True RUL
The 6th Device
Estimate of RUL
0
1000
3000
500
1000
1500
2000
2500
3000
3500
4000
Time
RUL prediction
95% credible interval
True RUL
The 10th Device
Estimate of RUL
0
1000
2000
3000
4000
500
1000
1500
2000
2500
3000
3500
4000
Time
RUL prediction
95% credible interval
True RUL

Conclusion

- Conjugate prior for the homogeneous gamma process is derived, and the properties of the prior are investigated.
- Three advanced algorithms (Gibbs sampling, DGS, and SIR) are proposed to simulate random numbers from the posterior distribution.
- The conjugate prior framework is extended to encompass the gamma process with heterogeneous effects.
- An innovative online algorithm is developed for simultaneous RUL prediction across multiple systems.

Thanks!